Proofs of the Grobman-Hartman theorems

We will reduce the proofs of Theorems 1 and 2 to one main step, and then we will complete the proof of that step. We let *id* denote the identity map on $\mathbf{R}^n$, and we let $|| \phi ||_0$ denote the C^0 norm of a mapping.

(A) MAIN STEP. *Suppose $L : \mathbf{R}^n \to \mathbf{R}^n$ is a hyperbolic linear map. There is an $\varepsilon > 0$ depending on L such that the following holds.*

If $\phi_1 : \mathbf{R}^n \to \mathbf{R}^n, \phi_2 : \mathbf{R}^n \to \mathbf{R}^n$ are Lipschitz maps such that

$$|| \phi_i ||_0 \leq \varepsilon \tag{1}$$

and

$$Lip(\phi_i) < \varepsilon, \tag{2}$$

then, there is a unique continuous map $h_{\phi_1\phi_2} : \mathbf{R}^n \to \mathbf{R}^n$ such that $h_{\phi_1\phi_2} - id$ is a bounded continuous map and

$$(L + \phi_1) \circ h_{\phi_1\phi_2} = h_{\phi_1\phi_2} \circ (L + \phi_2). \tag{3}$$

Assuming (A), let us prove Theorems 1 and 2.

Proof of Theorem 2.

We may assume $f = L + \phi$ where $L = D_0 f,\ Lip(\phi) < \varepsilon$, and $\phi(x) = 0$ for $| x | \geq \varepsilon$ and ε is as small as we wish. Let ζ denote the zero map ($\zeta(x) = 0$ for all x).

By (A), there are unique maps $h_{\phi\zeta}$ and $h_{\zeta\phi}$ of bounded distance from the identity such that

$$f h_{\phi\zeta} = h_{\phi\zeta} L \tag{4}$$

and

$$L h_{\zeta\phi} = h_{\zeta\phi} f. \tag{5}$$

This gives us $fh_{\phi\zeta}h_{\zeta\phi} = h_{\phi\zeta}Lh_{\zeta\phi} = h_{\phi\zeta}h_{\zeta\phi}f$.

So, $h_{\phi\zeta}h_{\zeta\phi}$ is a continuous map and $h_{\phi\zeta}h_{\zeta\phi} - id$ is bounded. By uniqueness of the solution h to $fh = hf$, we have $h_{\phi\zeta}h_{\zeta\phi} = id$. Similarly, $h_{\zeta\phi}h_{\phi\zeta}L = Lh_{\zeta\phi}h_{\phi\zeta}$. By uniqueness of the solutions of $hL = Lh$, we have $h_{\zeta\phi}h_{\phi\zeta} = id$.

Thus, $h_{\phi\zeta}$ is a homeomorphism and Theorem 2 is proved. □

Proof of Theorem 1.

This proof illustrates a general principle of the theory of linearizations. To linearize a vector field near a critical point x_0, it is sufficient to linearize its time one map near x_0.

We will reduce the proof of Theorem 1, to the statement we proved in Theorem 2.

We may assume that $X = D_0X + \phi$ with $\phi(x) = 0$ for $\mid x \mid \geq \varepsilon$ and $Lip(\phi) < \varepsilon$ with ε small. Let ψ_t be the time t map of X, and let η_t be the time t map of D_0X. Then, $\psi_1 = \eta_1 + \gamma$ where $\gamma = 0$ off a small neighborhood of 0, and $Lip(\gamma)$ is small. By the above proof of Theorem 2, we know there is a unique continuous map $h : \mathbf{R}^n \to \mathbf{R}^n$ such that $h - id$ is bounded and $h^{-1}\eta_1 h = \psi_1$ which is equivalent to $\eta_1 h = h\psi_1$ and $\eta_1 h\psi_{-1} = h$.

Let

$$H(x) = \int_0^1 (\eta_t h \psi_{-t})(x)\, dt. \tag{6}$$

Then, for $0 \leq s \leq 1$,

$$\eta_s H \psi_{-s} = \int_0^1 \eta_{s+t} h \psi_{-(t+s)}\, dt \quad \text{(since each } \eta_s \text{ is linear).}$$

Making the change of variable $u = s + t - 1$, we get the last integral equal to

$$\int_{s-1}^{s} \eta_{u+1} h \psi_{-1-u}\, du = \int_{s-1}^{0} \eta_{u+1} h \psi_{-1-u}\, du + \int_0^s \eta_u \eta_1 h \psi_{-1} \psi_{-u}\, du$$

Now set $v = u + 1$ in the first integral of the preceding equation and use the fact that $\eta_1 h \psi_{-1} = h$ in the second one.

We get the sum of integrals to be

$$\int_s^1 \eta_v h \psi_{-v}\, dv + \int_0^s \eta_u h \psi_{-u}\, du = H$$

.

Thus we have obtained that for $0 \leq s \leq 1$,

$$\eta_s H \psi_{-s} = H. \tag{7}$$

Thus, H conjugates each η_s to ψ_s. We would like to show that H is a homeomorphism.

Let us first show that H is continuous and $H - id$ is bounded.

Continuity is easy since if y is near x, the functions $t \to \eta_t h \psi_{-t}(x)$ and $t \to \eta_t h \psi_{-t}(y)$ are uniformly close.

Let us now see that $H - id$ is bounded.

We have

$$H(x) - x = \int_0^1 \eta_t h \psi_{-t}(x) dt - \int_0^1 x dt$$

For each $t \in [0,1]$, we have

$$\begin{aligned} \mid \eta_t h \psi_{-t}(x) - x \mid &= \mid \eta_t h \psi_{-t}(x) - \psi_t \psi_{-t}(x) \mid \\ &= \mid \eta_t h(u) - \psi_t(u) \mid \text{ for } u = \psi_{-t}(x) \\ &= \mid \eta_t h(u) - \eta_t(u) + \eta_t(u) - \psi_t(u) \mid \end{aligned}$$

But, $\eta_t(u) = \psi_t(u)$ for all $\mid t \mid \leq 1$ if $\mid u \mid$ is large.

Hence, if we let $L_0 = D_0 X$, then $\eta_t = e^{tL_0}$. Thus,

$$\mid \eta_t h \psi_{-t}(x) - x \mid \leq \sup_{0 \leq s \leq 1} \mid e^{sL_0} \mid \cdot \mid\mid h - id \mid\mid_0 + C_1$$

uniformly in t

where

$$C_1 = \sup_{\mid t \mid \leq 1, u \in \mathbf{R}^n} \mid \eta_t(u) - \psi_t(u) \mid .$$

This gives us that $H - id$ is bounded.

Now,

$$H - id \text{ is continuous and bounded }, \tag{8}$$

and, for $s = 1$,

$$\eta_1 H = H\psi_1 \tag{9}$$

But, h also satisfies (8) and (9), and we showed that there was a *unique* map satisfying these conditions. Thus, $H = h$, and the map h constructed for the time-one maps actually satisfies (7) for $0 \leq s \leq 1$.

We also want $\eta_s h \psi_{-s} = h$ for $-1 \leq s \leq 0$.

But, for $s \in [-1, 0]$, we have $-s \in [0, 1]$, so $\eta_{-s} h \psi_{-(-s)} = h$. Now compose on the left with η_s and on the right with ψ_{-s} and we get $h = \eta_s h \psi_{-s}$.

This proves Theorem 1. $\square$