# The Grobman-Hartman theorem

Now that we have studied the structure of solutions to linear differential equations in general, we wish to use that theory to study the local structure of the solutions to non-linear systems. If $X$ is a $C^r$ vector field, $r \geq 1$, defined in an open set $U \subset \mathbf{R}^n$, and $x_0 \in U$ is a non-singular point (i.e., $X(x_0) \neq 0$), then we have seen that there is a $C^r$ change of coordinates which takes solutions near $x_0$ to straight lines. Thus, it remains to describe the solutions near a critical point. If the derivative $A = DX_{x_0}$ of $X$ at $x_0$ has eigenvalues with real parts different from zero, we will see, that after a continuous change of coordinates, the structure of solutions of $X$ near $x_0$ is the same as that of the linear system $\dot{y} = Ay$ near 0.

We now make the relevant definitions.

Let $X$ be a $C^r$ vector field as above with $r \geq 1$ with a critical point at $x_0$ (i.e., $X(x_0) = 0$). Let $A$ be the derivative of $X$ at $x_0$. Thus, $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map whose matrix in the standard coordinates on $\mathbf{R}^n$ is the Jacobian matrix of $X$ at $x_0$.

**Definition.** The critical point $x_0$ of $X$ is called *hyperbolic* if the eigenvalues of $A$ all have non-zero real parts.

If $X$ is a $C^r$ vector field, recall that the *local flow* of $X$ near $x_0$ is the function $\eta(t, x)$ defined in a neighborhood $V$ of $(0, x_0)$ in $\mathbf{R}^{n+1}$ such that

1. $\eta(0, x) = x$ for $(0, x) \in V$
2. $t \to \eta(t, x)$ is a solution to the differential equation $\dot{x} = X(x)$ defined in a neighborhood of $t = 0$.

We also use the notation $\eta_t$ for the local flow $\eta(t, x)$. We sometimes call $\eta_t$ the local flow of the differential equation $\dot{x} = X(x)$ as well. We will also use the term *integral curve* of the vector field $X$ for a solution curve.

**Definition.** A linear map $L : \mathbf{R}^N \to \mathbf{R}^N$ is called *hyperbolic* if its (possibly complex) eigenvalues have norm different from one.

**Examples:**

1. $L$ is the map induced by the $2 \times 2$ matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
2. $L = e^A$ where $A$ is a linear map whose eigenvalues have non-zero real parts.

If $L$ is a hyperbolic linear map of $\mathbf{R}^N$, then there is a direct sum decomposition $\mathbf{R^N} = E^s \oplus E^u$ such that

1. $L(E^s) = E^s$ and $L(E^u) = E^u$
2. the eigenvalues of $L \mid E^s$ have norm $< 1$ and those of $L \mid E^u$ have norm $> 1$

**Definition** Let $X$ be a $C^r$ vector field defined in a neighborhood of $x_0$ in $\mathbf{R}^N$ having $x_0$ as a critical point. Let $DX_{x_0}$ be the derivative of $X$ at $x_0$. A $C^0$ linearization of $X$ near $x_0$ is a homeomorphism $h$ from a neighborhood $U$ of $x_0$ in $\mathbf{R}^N$ to a neighborhood of 0 such that if $\eta_t$ is the local flow of $X$ near $x_0$, then $h\eta_t h^{-1}$ is the local flow of the linear differential equation $\dot{y} = DX_{x_0} \cdot y$ near 0.

One may similarly define $C^k$ linearizations of a $C^r$ vector field $X$ for $1 \leq k \leq r$ by requiring that $h$ be a $C^k$ diffeomorphism from a neighborhood of $x_0$ to a neighborhood of 0.

**Theorem 1.** (Grobman-Hartman). *Suppose $x_0$ is a hyperbolic critical point of the $C^1$ vector field $X$. Then $X$ has a $C^0$ linearization near $x_0$.*

**Remark.** For smooth linearizations, one has the following result.

**Theorem.** *Suppose that $L$ is linear map on $R^n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $L$. For each positive integer $k$, there is a positive integer $N(k)$ with the following property. Suppose that for each $1 \leq i \leq n$ and each $n-$tuple $(m_1, m_2, \ldots, m_n)$ of non-negative integers satisfying $2 \leq \sum_{1 \leq j \leq n} m_j \leq N(k)$, we have $\lambda_i \neq \sum_{1 \leq j \leq n} m_j \lambda_j$.*

*Then, any $C^{N(k)}$ vector field $X$ with $X(x_0) = 0$ and $DX_{x_0} = L$ has a local $C^k$ linearization near $x_0$.*

Note that as a corollary of the Grobman-Hartman theorem, we have

**Corollary.** *Let $x_0$ be a hyperbolic critical point of a $C^1$ vector field $X$ in $\mathbf{R}^n$. If all the eigenvalues of the derivative $L = DX_{x_0}$ have negative real parts, then $x_0$ is asymptotically stable. If $L$ has at least one eigenvalue with positive real part, then $x_0$ is unstable.*

We will proceed toward the proof of Theorem 1. Note that we may assume that both $X$ and $L$ have local flows defined for $\mid t \mid \leq 1$.

In the course of the proof, it will be necessary to first linearize the time-one map $\eta_1$ of $X$ near $x_0$. So, we first study the relevant linearization theorem for local diffeomorphisms.

**Definition.** Let $f$ be a $C^1$ diffeomorphism from a neighborhood $U$ of $x_0$ in $\mathbf{R}^n$ into $\mathbf{R}^n$ with $f(x_0) = x_0$. The fixed point $x_0$ is called *hyperbolic* if all the eigenvalues of $Df_{x_0}$ have absolute values with norm different from one; i.e, $Df_0$ is a hyperbolic linear map.

**Theorem 2.**(Grobman-Hartman theorem for local diffeomorphisms). *Suppose $x_0$ is a hyperbolic fixed point of the local $C^1$ diffeomorphism $f$ defined on a neighborhood $U$ of $x_0$ in $\mathbf{R}^n$. Let $L = Df_{x_0}$. There is a neighborhood $U_1 \subseteq U$ of $x_0$ and a homeomorphism $h$ from $U_1$ into $\mathbf{R}^n$ such that $h(x_0) = 0$ and $hf(x) = Lh(x)$ for $x \in U_1 \bigcap f^{-1}U_1$.*

Note that an equivalent formulation of

$$hf(x) = Lh(x) \text{ for } x \in U_1 \bigcap f^{-1}U_1$$

is

$$hfh^{-1}(y) = L(y) \text{ for } h^{-1}(y) \in U_1 \bigcap f^{-1}U_1$$

so the formulas in both theorems are analogous.
*Remark.*

1. The proofs we will give of the above theorems are valid if $\mathbf{R}^n$ is replaced by a Banach space.

2. A map $h$ as in Theorem 2 is called a $C^0$ linearization of $f$. One may define $C^k$ linearizations analogously for $k \geq 1$.

*Definition.* Let $U$ be an open subset of $\mathbf{R}^n$ and let $g : U \to R^n$ be a mapping. We say $g$ is *Lipschitz* (or *Lipschitz continuous*) if there is a constant $K > 0$ such that

$$\sup_{x \neq y} \frac{\mid g(x) - g(y) \mid}{\mid x - y \mid} \leq K < \infty.$$

When $g$ is Lipschitz, we let $Lip(g) = \sup_{x \neq y} \dfrac{\mid g(x) - g(y) \mid}{\mid x - y \mid}$ and we call it the *Lipschitz constant* of $g$.

Note that if $g$ is $C^1$ and $M = \sup_x \mid D_x g \mid$, then $g$ is Lipschitz and $Lip(g) = M$. That is, the maximum of the norms of the derivatives of a $C^1$ map $g$ equals the Lipschitz constant of $g$.

We will develop some machinery to prove Theorem 2. Then we will prove Theorem 1.

Let us first note that, replacing $f$ by $x \to f(x + x_0) - x_0$, we may assume $x_0 = 0$.

**Lemma 3.** There is a $C^\infty$ function $\alpha : \mathbf{R} \to [0, 1]$ such that

1. $\alpha(u) = 1$ for $u \leq \frac{1}{2}$.

2. $\alpha(u) = 0$ for $u \geq 1$.

**Proof.**

Let

$$\phi(u) = \begin{cases} exp(-\frac{1}{(\frac{1}{2} - u)(u-1)}) & \text{for } \frac{1}{2} < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, $\phi$ is $C^\infty$.

Let

$$\psi(u) = \frac{\int_{-\infty}^{u} \phi(s)\, ds}{\int_{-\infty}^{1} \phi(s)\, ds}$$

Then, $\psi$ is $C^\infty$ and

$$\psi(u) = \begin{cases} 0 & \text{for } u \leq \frac{1}{2} \\ 1 & \text{for } u \geq 1 \end{cases}$$

and $\psi(u) \in [0, 1]$ for all $u$.

Let $\alpha(u) = 1 - \psi(u)$. Then, $\alpha$ has the required properties. (Details left as an exercise.)

Let $f$ be as in the statement of Theorem 2. Our next lemma will show that we may assume there is a $\delta > 0$ such that $f$ is defined on all of $\mathbf{R}^n$, $f(x) = L(x)$ for $\mid x \mid \geq \delta$ and $Lip(f - L)$ (on all of $\mathbf{R}^n$) is small.

**Lemma 4.** Let $\varepsilon > 0$. There are a $\delta > 0$ and a $C^1$ diffeomorphism $f_1 : \mathbf{R}^n \to \mathbf{R}^n$ such that

1. $f_1(x) = L(x)$ for $\mid x \mid \geq \delta$
2. $f_1(x) = f(x)$ for $\mid x \mid \leq \frac{\delta}{2}$
3. $Lip(f_1 - L) < \varepsilon$ and $\|\, f_1 - L \,\|_0 < \varepsilon$.

(Here, $\|\, f_1 - L \,\|_0 = \sup_{x \in \mathbf{R}^n} \mid f_1(x) - L(x) \mid$.)

*Proof.* Let $\varepsilon_1 \in (0,1)$, and let $\delta_1 \in (0,1)$ be small enough so that

(a) $f$ is defined for $\mid x \mid \leq \delta_1$

(b) $\|\, D_x(f - L) \,\| < \varepsilon_1$
and

(c) $\mid f(x) - L(x) \mid < \varepsilon_1 \delta_1$ for $\mid x \mid \leq \delta_1$

Let $\alpha$ be as in Lemma 3, and let $K = \sup_{u \in \mathbf{R}} \mid \alpha'(u) \mid$.
Let $\gamma(x) = \alpha(\frac{\mid x \mid}{\delta_1})$. Note that $\mid D_x \gamma \mid \leq \frac{K}{\delta_1}$ for all $x$.

Now,

$$\gamma(x) = \begin{cases} 1 & \text{for } \mid x \mid \leq \frac{\delta_1}{2} \\ 0 & \text{for } \mid x \mid \geq \delta_1 \end{cases}$$

Let

$$\begin{aligned} f_1(x) &= \gamma(x) f(x) + (1 - \gamma(x)) L(x) \\ &= L(x) + \gamma(x)(f(x) - L(x)) \end{aligned}$$

Note that $f_1$ is the $\gamma-$average of $f$ and $L$.
Now, $(f_1 - L)(x) = \gamma(x)(f(x) - L(x))$, so

$$\begin{aligned}|| f_1 - L ||_0 &= \sup_x | \gamma(x)(f(x) - L(x)) | \\ &\leq \sup_{| x | \leq \delta_1} | f(x) - L(x) | \leq \quad \varepsilon_1\end{aligned}$$

Also,

$$\begin{aligned}|| D_x(f_1 - L) || &= | D_x\gamma \cdot (f(x) - L(x)) + \gamma(x)(D_x f - L) | \\ &\leq | D_x\gamma || f(x) - L(x) | + || D_x f - L ||_0 \\ &\leq \frac{K}{\delta_1}\varepsilon_1\delta_1 + \varepsilon_1\end{aligned}$$

Note that we use the notation $D_x\gamma \cdot (f(x) - L(x))$ for the map $v \to D_x\gamma(v)(f(x) - L(x))$.

Now, given $\varepsilon \in (0,1)$, choose $\varepsilon_1 \in (0,1)$ small enough so that $\max(\varepsilon_1, K\varepsilon_1 + \varepsilon_1) < \varepsilon$. $\square$