Remark. The Variation of Parameters formula $\Phi v' = h$ for the function v giving the particular solution $x_p(t) = \Phi(t)v(t)$ should be treated as a system of linear equations with unknown vector v'. Thus, to find the solution $x_p(t)$, one simply solves this system for $v'(t)$ and integrates to find $v(t)$.

Some examples

We now consider some examples of linear differential equations with constant coefficients

1.

$$\begin{aligned} x' &= -2x \\ y' &= y \end{aligned}$$

Here the matrix A is

$$\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix},$$

the eigenvalues are $-2, 1$, and the general solution is

$$\bar{x}(t) = e^{-2t} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + e^{t} \begin{pmatrix} 0 \\ c_2 \end{pmatrix}$$

The critical point 0 is called a *saddle*.

The orbits near 0 are depicted in the figure below.

2.

$$\begin{aligned} x' &= 2x - y \\ y' &= x + y \end{aligned}$$

The matrix A is

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix},$$

The characteristic polynomial is $\lambda^2 - 3\lambda + 3$, and the eigenvalues are

$$\lambda = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$$

Letting $\lambda = \frac{3}{2} + i\frac{\sqrt{3}}{2}$, we have the matrix equation

$$(A - \lambda I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $(2-\lambda)v_1 = v_2$, so that a complex eigenvalue is $(v_1, v_2) = (1, 2-\lambda)$.

We get a complex solution of the form

$$\begin{aligned}
\bar{x}_c(t) &= e^{\lambda t}\begin{pmatrix} 1 \\ 2-\lambda \end{pmatrix} \\
&= e^{(\frac{3}{2}+i\frac{\sqrt{3}}{2}t)}\begin{pmatrix} 1 \\ \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} \\
&= e^{(\frac{3}{2}+i\frac{\sqrt{3}}{2}t)}\left(\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + i\begin{bmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}\right)
\end{aligned}$$

The real and imaginary parts of this are

$$Re = e^{\frac{3}{2}t}\left(cos(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} - sin(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}\right)$$

$$Im = e^{\frac{3}{2}t}\left(cos(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix} + sin(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}\right)$$

See the figure.

3.

$$\begin{aligned} x' &= 2x \\ y' &= 2y \end{aligned}$$

The matrix A is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

The characteristic polynomial is $(\lambda - 2)^2$, and the only eigenvalue is 2. The general solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4.

$$\begin{aligned} x' &= x + y \\ y' &= y \end{aligned}$$

The matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We have

$$\begin{aligned} e^{tA} &= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \left[I + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] \\ &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \end{aligned}$$

so the general solution is

$$\begin{aligned} \mathbf{x}(t) &= e^{tA} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{bmatrix} c_1 e^t + c_2 t e^t \\ c_2 e^t \end{bmatrix} \end{aligned}$$

5.

$$\begin{aligned} x' &= 3x + 11y + 5z \\ y' &= -x - y - z \\ z' &= 2x + z \end{aligned}$$

The matrix is

$$A = \begin{pmatrix} 3 & 11 & 5 \\ -1 & -1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \lambda^3 - 3\lambda^2 + 4 = (\lambda - 2)^2(\lambda + 1).$$

Eigenvalue $\lambda = 2$:
Let $N = A - 2I$. Then,

$$N = \begin{bmatrix} 1 & 11 & 5 \\ -1 & -3 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

$rank(N) = 2.$

$$N^2 = \begin{bmatrix} 0 & -22 & -11 \\ 0 & -2 & -1 \\ 0 & 22 & 11 \end{bmatrix}$$

$rank(N)^2 = 1.$
The vector

$$v = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is in $ker(N)$.

The vector

$$w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

satisfies $Nw = v$.

We get two linearly independent solutions in $ker(N^2)$ by

$$e^{2t}v, e^{2t}(w + tNw)$$

the eigenvalue $\lambda = -1$.

Let $N = A + I$.

Then,

$$N = \begin{bmatrix} 4 & 11 & 5 \\ -1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$$

Then, $rank(N) = 2$, and $ker(N)$ is one-dimensional.

The vector

$$v = \begin{pmatrix} -1 \\ -\frac{1}{11} \\ 1 \end{pmatrix}$$

is in the kernel of N, so is an eigenvector for A associated to $\lambda = -1$.

A fundamental set of solutions, then, is the set

$$e^{2t}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, e^{2t}(I + tN)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e^{-t}\begin{pmatrix} -1 \\ -\frac{1}{11} \\ 1 \end{pmatrix}$$

where

$$N = \begin{bmatrix} 1 & 11 & 5 \\ -1 & -3 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

Definition. We say that an $n \times n$ matrix A is *hyperbolic* if all of its (possibly complex) eigenvalues have non-zero real parts.

Proposition. *Let $gl(n, \mathbf{R})$ denote the set of $n \times n$ real matrices. The set of hyperbolic elements in $gl(n, \mathbf{R})$ is dense and open in $gl(n, \mathbf{R})$.*

Proof.

Density:

Given a matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, let $B = A + \epsilon I$ for small positive ϵ. The eigenvalues of B are $\lambda_j + \epsilon$. So, if $\epsilon > 0$ is sufficiently small and positive, then B is near A and hyperbolic.

Openness:

Suppose that B is a hyperbolic matrix with characteristic polynomial

$$p(\lambda) = \sum_{j=0}^{n-1} a_j \lambda^j + \lambda^n.$$

Let $\lambda_1, \ldots, \lambda_n$ be the roots of $p(\lambda)$.

Let

$$\delta = \min_{j \neq k} \mid \lambda_j - \lambda_k \mid,$$

and let $0 < \epsilon < \delta$.

For a sequence $b_j, 0 \leq j < n$, let

$$q(\lambda) = \sum_{j=0}^{n-1} b_j \lambda^j + \lambda^n$$

Let $K > 0$ be such that for $\mid z \mid > K$, and $\mid b_j - a_j \mid < 1$, we have $\mid q(z) \mid > 1$.

Let $\epsilon > 0$ be small enough so that each open ball $B_\epsilon(\lambda_j)$ is disjoint from the imaginary axis in $\mathbf{C}$.

The function $p(z)$ is non-zero on the compact set

$$E = (\{z : \mid z \mid \leq K\}) \bigcap (\bigcup_j \mathbf{C} \setminus B_\epsilon(\lambda_j))$$

Since $p(z)$ is continuous and non-zero on the compact set E, there is a constant $c > 0$ such that $\mid p(z) \mid > c$ for all $z \in E$. If q is an $n - th$ degree polynomial whose coeficients are close to those of p, then q has the properties that

1. $\mid q(z) \mid > \frac{c}{2}$ for $z \in E$
2. $\mid q(z) \mid > 1$ for $\mid z \mid > K$

This implies that the roots of q lie in $\bigcup_j B_\epsilon(\lambda_j)$. Now, if C is a matrix whose entries are close to those of B, the coeficients of the characteristic polynomial of C are close to those of B. Hence, C will be hyperbolic by the above observations. QED

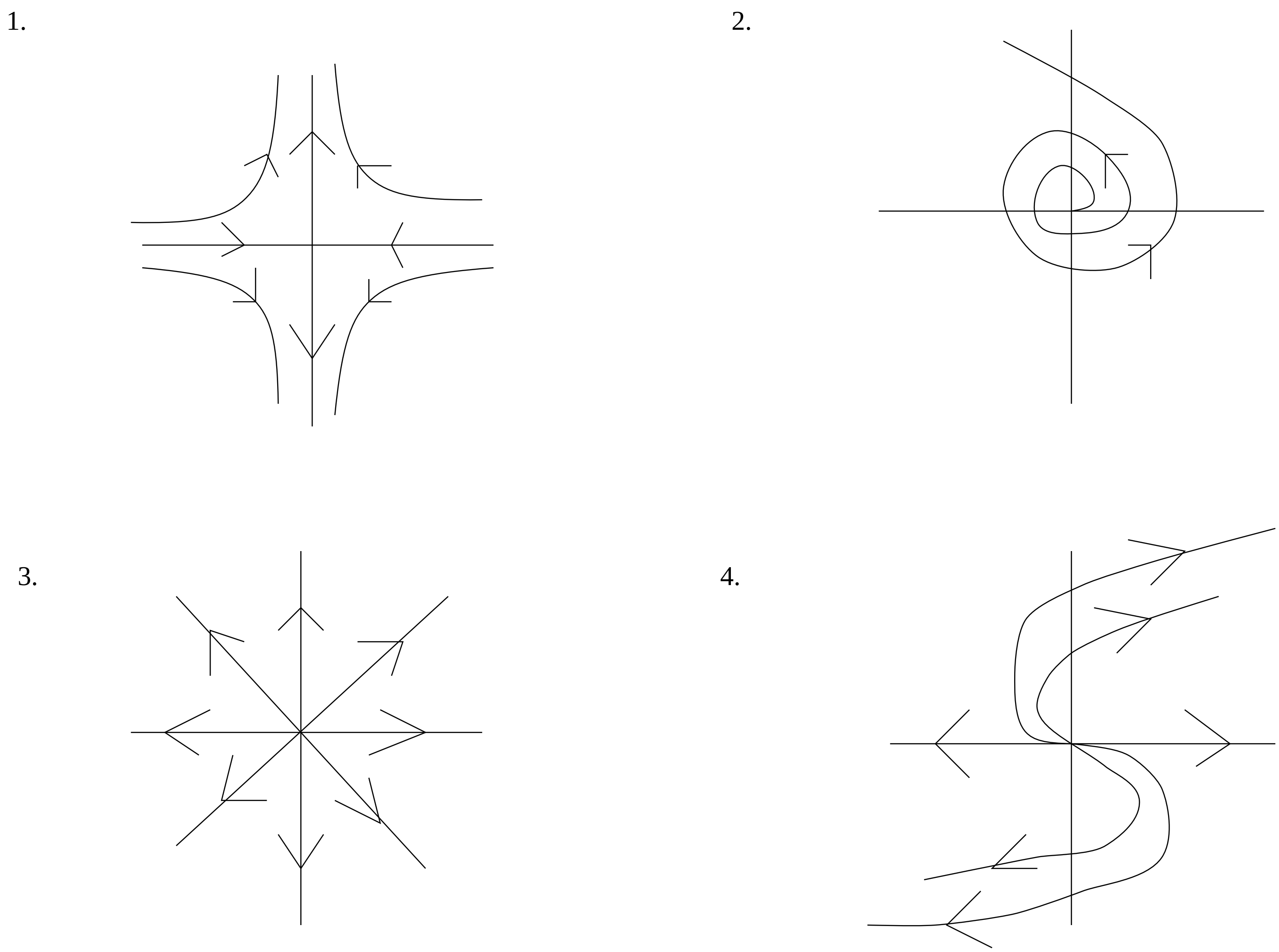

Figure 1: Figure for Examples 1-4