Alternate treatment of Section 11

For a Jordan curve γ, let us write *int* γ for the bounded interior region of the complement of γ.

Let f be a planar C^1 vector field with an isolated critical point x_0. and let γ be a positively oriented C^1 Jordan curve so that the only critical point of f in *int* $\gamma \bigcup \gamma$ is x_0. Let $\phi(t,x)$ be the local flow of f. A point $y \in \gamma$ at which f is tangent to γ is called an *exterior tangency* if there is an $\epsilon > 0$ such that $\phi(t,x)$ is in the exterior of γ for $t \in (-\epsilon, \epsilon) \setminus \{0\}$. Similarly, the point y of tangency is an *interior tangency* if there is an $\epsilon > 0$ such that $\phi(t,x) \in$ *int* γ for $t \in (-\epsilon, \epsilon) \setminus \{0\}$.

In general, a tangency may be neither exterior nor interior.

Suppose f has only finitely many tangencies with γ and they are all interior or exterior. An interval I in γ between two such tangencies will be called

1. *interior* if its boundary points are both interior points
2. *exterior* if its boundary points are both exterior points
3. *neutral* if its boundary points consist of one interior and one exterior tangency.

Theorem. *Suppose that f and γ are as above and there are only finitely many points of tangency of f and γ and all of these tangencies are exterior or interior. Let N_i be the number of interior tangencies and N_e be the number of exterior tangencies. Then,*

$$Ind(x_0, f) = 1 + \frac{1}{2}(N_i - N_e). \tag{1}$$

Also, if $\tilde{N}_i$ is the number of interior intervals, and $\tilde{N}_e$ is the number of exterior intervals, then

$$Ind(x_0, f) = 1 + \frac{1}{2}(\tilde{N}_i - \tilde{N}_e). \tag{2}$$

Proof.

Let's prove the second statement first.

Let $y_0, y_1, \ldots, y_n$ be the tangencies of f at the curve γ where γ is as in the statement of the theorem and $y_0 = y_n$.

For $1 \leq m \leq n$, let $var(y_0, y_m, \gamma)$ be the angular variation of the tangent vector to γ from y_0 to y_m, and let $var(y_0.y_m, f)$ be the angular variation of f from y_0 to y_m. Let

$$\beta(m) = var(y_0.y_m, f) - var(y_0, y_m, \gamma).$$

Note that if the interval $[y_m, y_{m+1}]$ is

- interior: then $\beta(m+1) - \beta(m) = \pi$
- exterior: then $\beta(m+1) - \beta(m) = -\pi$
- neutral: then $\beta(m+1) - \beta(m) = 0$.

Hence, if γ' denotes the tangent vector field on γ, then

$$\begin{aligned} 2\pi Ind(f, \gamma) &= 2\pi Ind(\gamma', \gamma) + \beta(n) \\ &= 2\pi Ind(\gamma', \gamma) + \sum_{i=0}^{n-1} \beta(i+1) - \beta(i) \\ &= 2\pi Ind(\gamma', \gamma) + \pi(\tilde{N}_i - \tilde{N}_e). \end{aligned}$$

Dividing both sides by 2π gives (2).

Now we turn to (1).

First observe that if f has only internal or external tangencies, then, $N_i = \tilde{N}_i$ and $N_e = \tilde{N}_e$, so the result holds by (2).

So, we may assume that there are tangencies of both types, and hence at least one neutral interval.

We say that f is *internal* on an open interval $I_i = (y_i, y_{i+1})$ if it points into the interior of γ and otherwise we say that f is *exterior* on I_i.

Note that in going across a tangency from one interval I_i to I_{i+1}, f alternates from interior to exterior or vice-versa.

We want to prove (1) by induction on the number of tangencies.

Note that the statement only depends on the structure of f *on* the curve γ, and not on how f behaves at points off γ.

Let $I_i = (y_i, y_{i+1})$ be a neutral interval. We squeeze I_i to a point bringing its boundary points together, say to a point p. Doing this we change f, say to f_1 and γ, say to γ_1. We can do this so that we reduce the number of tangencies by one and create a tangency at p which looks topologically like a point of "cubic contact." After that we can turn f_1 slightly near p and remove the tangency at p entirely without introducing any new tangencies. This entire procedure can be done without changing the index of f and without changing the difference $\tilde{N}_i - \tilde{N}_e$. Thus we will have produced a new vector field f_2 on a new curve γ_1 such that

(a) f_2 has only interior and exterior tangencies with γ_1,

(b) $j_{f_1}(\gamma_1) = j_f(\gamma)$, and

(c) $\tilde{N}_i(f_2) - \tilde{N}_e(f_2) = \tilde{N}_i(f) - \tilde{N}_e(f)$.

By induction, we get our result.

Note also that we could continue this procedure and remove all neutral intervals without changing either side of (1). QED

Let us continue with our assumptions that f is a C^1 planar vector field with an isolated critical point at x_0, and suppose γ is a small positively oriented C^1 Jordan curve γ with $x_0 \in int\ \gamma$ such that f does not vanish on $int\ \gamma \bigcup \gamma$ except at x_0 .

Let $\phi(t,x)$ be the local flow of f. Recall we are assuming that $\phi(t,x)$ is defined for all t.

A solution $\phi(t,x)$ through a point $x \in \gamma$ is called a *positive null solution relative to* γ if it satisfies the following condition.

- $\phi(t,x)$ is defined for all $t \geq 0$, there is a $t_1 > 0$ such that $\phi(t,x) \in int\ \gamma$ for all $t > t_1$ and $\phi(t,x) \to x_0$ as $t \to \infty$.

The solution $\phi(t,x)$ is called a *negative null solution relative to* γ if $\phi(-t,x)$ is a positive null solution for $-f$.

A *null solution relative to* γ is either a positive or negative null solution relative to γ. When γ is understood, we refer simply to null solutions, positive null solutions, or negative null solutions. We also speak of null, positive null, and negative null orbits for the set of points along the images of such solutions.

A solution $\phi(t,x)$ is called *elliptic* if it is both a positive and negative null solution.

A *base interval* in γ is an open interval U in γ whose boundary points belong to null orbits.

The base interval U is a *parabolic* interval if all of its points belong to positive null orbits or all of its points belong to negative null orbits. The base interval is an *elliptic* interval if all of its points belong to elliptic orbits. A base interval which is called *hyperbolic* if none of its points belongs to a null orbit. Suppose that there is at least one null solution. Then, it can be shown that, for any point x in a hyperbolic interval there are times $t_2(x) > 0, t_1(x) < 0$ such that $\phi(t,x)$ is exterior to γ for all $t \notin (t_1(x), t_2(x))$.

The union of the set of orbits of points belonging to a parabolic, elliptic, or hyperbolic base interval will be called, respectively, a parabolic, elliptic, or hyperbolic sector.

The following theorem can be proved more or less like the previous one.

Theorem. *Suppose that f, x_0, and γ are as above and that there are finitely many points $y_0, y_1, \ldots, y_n$ on γ whose orbits are base solutions. Assume that the points in $\gamma \setminus \{y_0, \ldots, y_n\}$ belong only to elliptic, hyperbolic, and parabolic intervals. Let N_{ell}, N_{hyp} denote, respectively the number of elliptic, hyperbolic intervals. Then,*

$$Ind(f, x_0) = 1 + \frac{1}{2}(N_{ell} - N_{hyp})$$

For more general information about these topics, see (P. Hartman, *Ordinary Differential Equations*, 1973, Chapter 7).

Note that Hartman's definitions of sectors are slightly different than the ones given here.

Index of isolated critial point in $\mathbf{R}^{n+1}$

We now briefly discuss the concept of index for an isolated critical point of an autonomous vector field in $\mathbf{R}^{n+1}$ for arbitrary $n \geq 1$. This involves the notion of degree of a continuous self-mapping of the $n-$sphere. There are many equivalent ways to define this notion. The simplest involves homology theory. Since we do not assume knowledge of this theory, we will give a definition in terms of integration on the $n-$sphere S^n.

Degree of maps of S^n using integration theory

We refer to standard texts; e.g. for the basic concepts of differential forms and integration of manifolds, see (F. Warner, Foundations of differentiable manifolds and Lie groups, Springer, 1983.)

Let D^n be the open unit ball in $\mathbf{R}^n$, and let S^n be the (unit) $n-$sphere; i.e., the set of vectors in $\mathbf{R}^{n+1}$ whose distance from the origin is exactly equal to 1. Write $x = (x_1, x_2, \ldots, x_n)$ for coordinates in $\mathbf{R}^n$. Let $r \geq 1$. A C^r coordinate parametrization in S^n is a pair (U, ϕ) where ϕ is a 1-1 C^r map from D^n into $\mathbf{R}^{n+1}$ such that

1. $\phi(x) \in S^n$ for all $x \in D^n$
2. The columns of the Jacobian matrix of ϕ, $\frac{\partial \phi}{\partial x_i}$ are linearly independent in $\mathbf{R}^{n+1}$.
3. $U = image(\phi)$ and U is an open subset of S^n.

If (U, ϕ) and (V, ψ) are two C^r coordinate parametrizations in S^n, and $U \bigcap V \neq \emptyset$, then the map $\psi^{-1} \circ \phi$ is a C^r diffeomorphism from the open set $\phi^{-1}(U \bigcap V)$ to $\psi(U \bigcap V)$.

If (U, ϕ) is a coordinate parametrization in S^n, we sometimes call the pair (U, ϕ^{-1}) a coordinate chart. Thus, the maps in coordinate parametrizations go from D^n into S^n, and the maps in coordinate charts go from open subsets of S^n to D^n. Coordinate charts (U, η), (V, γ) have the property of differentiability on overlaps in the sense that $\eta \circ \gamma^{-1}$ is C^r where it is defined.

A C^k map $f : S^n \to \mathbf{R}^{n+1}$ is a continuous map from S^n to $\mathbf{R}^{n+1}$ such that, for each C^k coordinate chart (U, ϕ) in S^n, the composition $f \circ \phi^{-1}$ is a C^k map from D^n into $\mathbf{R}^{n+1}$.

A C^k tangent vector field X on S^n is a C^k map $X : S^n \to \mathbf{R}^{n+1}$ such that, for each $x \in S^n$, $X(x)$ is tangent to S^n at x. A positively oriented C^k orthonormal $n-$frame field on S^n is an $n-$tuple $(v_1, \ldots, v_n)$ of C^k tangent vector fields on S^n with the property that, for each $x \in S^n$, the set of vectors $\{v_1(x), \ldots, v_n(x)\}$ is an orthonormal basis for the tangent space to S^n at x, and the determinant $Det(v_1(x), v_2(x), \ldots, v_n(x), x)$ is positive.

There is a C^∞ $n-$form on S^n which gives the value 1 to each positively oriented orthonormal $n-$frame field on S^n. We call this form the *unit volume form* and write it dv.

Using this form, one can define integration for continuous functions $g : S^n \to \mathbf{R}$ by the formula

$$\int_{S^n} g(x) = \int_{S^n} g(x) dv(x)$$

The volume of the whole sphere is then the constant $vol(S^n)$ which is the integral of the constant function whose value is 1 for each point of S^n.

Given a C^1 map $f : S^n \to S^n$, one can pull back the form dv to a form $f^\star dv$, and integrate this form over S^n. One gets an integer multiple of $vol(S^n)$ and this integer is called the *degree* of the map f. We write $deg(f)$ for this integer.

Thus, we have

$$\int_{S^n} f^\star dv = deg(f) vol(S^n)$$

Degree of maps of S^n using regular values

Write the $n-$sphere as

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : \sum_i x_i^2 = 1\}$$

We cover the points on S^n for which $x_{n+1} \neq 0$ by the two open hemispheres

$$U_+ = \{(x_1, \ldots, x_{n+1}) \in S^n : x_{n+1} = \sqrt{1 - \sum_{1 \le i \le n} x_i^2}\},$$

$$U_- = \{(x_1, \ldots, x_{n+1}) \in S^n : x_{n+1} = -\sqrt{1 - \sum_{1 \le i \le n} x_i^2}\}.$$

Define the maps $\phi_\pm : D^n \to S^n$ by

$$\phi_+(u_1, u_2, \ldots, u_n) = (u_1, u_2, u_3, \ldots, u_n, \sqrt{1 - \sum_{1 \le i \le n} u_i^2}),$$

and

$$\phi_-(u_1, u_2, \ldots, x_n) = (u_2, u_1, u_3, \ldots, u_n, -\sqrt{1 - \sum_{1 \le i \le n} u_i^2}).$$

These are coordinate parametrizaions of U_+, U_-, respectively.

Note the interchange of the two first coordinates u_1, u_2 in the definition of ϕ_-. This is to insure that the normal vectors to S^n induced by these coordinate parametrizations point outward on S^n. Note that there is a "cross-product" of n vectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^{n+1}$ defined by a slight modification of the usual determinant rule.

The *outward* normal vector $N(x)$ at some point $x \in S^n$ is the normal vector such that the dot product $N(x) \cdot x$ is positive.

The usual determinant rule for obtaining the outward normal to the plane spanned by the non-collinear vectors v_1, v_2 in $\mathbf{R}^3$ is to write the matrix A whose second and third row vectors are $v_1 = (v_{11}, v_{12}, v_{13}), v_2 = v_{21}, v_{22}, v_{23})$, respectively, and to put the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ across the top as

$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix} \tag{3}$$

Then, one gets the coordinates (relative to the standard basis of $\mathbf{R}^3$) of the normal vector by *expanding* by minors across the top row.

In the case of n linearly independent vectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^{n+1}$, let $V = sp(v_1, \ldots, v_n)$ be the linear subspace spanned by the $v_i's$. To get a normal vector to V by a rule like the one in (3) one has to put the basis vectors on the bottom of the analogous matrix A instead of the top as in

$$\begin{pmatrix} v_{11} & v_{12} & \ldots & v_{1,n+1} \\ v_{21} & v_{22} & \ldots & v_{2,n+1} \\ & \vdots & & \\ \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_{n+1} \end{pmatrix} \tag{4}$$

Looking at the top point $(0, 0, \ldots, 1)$ in S^n shows that, using the top row for the $\mathbf{e}_i's$ will result in an inward normal to S^n when *n is odd.*

Next, let $f : S^n \to S^n$ be a C^1 map. If f is not onto, we simply define $deg(f) = 0$.

Now, we assume that f is onto. A point $y \in S^n$ is a *regular value* of f if, for every $x \in f^{-1}(y)$ and any coordinate chart (U, ψ) with $x \in U$, the derivative of $f \circ \psi^{-1}$ has rank n at x.

It is a theorem (Sard's theorem) that almost all points of S^n are regular values. This means that the complement of the set of regular values has zero volume (Lebesgue measure) in S^n.

If y is a regular value of f, it follows that the set $f^{-1}(y) = E(y)$ is a finite set.

From Sard's theorem, it follows that there is a regular value $y \in U_+ \bigcup U_-$ whose preimage $f^{-1}(y) = E(y)$ is also in $U_+ \bigcup U_-$.

Given point $x \in E(y) \bigcap U_i \bigcap f^{-1}U_j$, with $i = \pm$ and $j = \pm$, we say that f *preserves orientation at* x if the determinant of the derivative of $\phi_j^{-1} f \phi_i$ at $\phi_i^{-1}(x)$ is positive.

Define $sgn(x) = 1$ if f preserves orientation at x, and $sgn(x) = -1$ is f reverses orientation at x.

For a regular value y of f as above, define the *degree of* f *at* y to be

$$deg(f, y) = \sum_{x \in f^{-1}(y)} sgn(x).$$

It is a fact that this number does not depend on which regular value one chooses. So, we define

$$deg(f) = deg(f, y)$$

for any regular value y as above.

We remark that, in general one could cover S^n by coordinate charts (V_i, ψ_i) such that $det(D(\psi_i \circ \psi_j^{-1})) > 0$ where defined, and do the construction above. This is usually called an *orientation* of S^n. We chose to use the sets U_+, U_- which *almost cover* S^n because they are familiar from elementary calculus.

Index of C^1 vector fields in $\mathbf{R}^n$

Now, let f be a C^1 vector field in $\mathbf{R}^n$ with an isolated critical point x_0. One defines the index of f at x_0, $Ind(f, x_0)$ in the following way.

Let S_ϵ be a small $(n-1)-$sphere of radius ϵ centered at x_0, and assume that $f(x) \neq 0$ for all x with $0 < | x - x_0 | \leq \epsilon$. Use f on S_ϵ to define a map $\bar{f} : S^{n-1} \to S^{n-1}$ by the formula

$$\bar{f}(y) = \frac{f(x_0 + \epsilon y)}{| f(x_0 + \epsilon y) |}$$

Then, we define

$$Ind(f, x_0) = deg(\bar{f})$$

One can show that the definition is independent of the choice of small $(n-1)$-sphere S_ϵ containing x_0 in its interior. It actually can be defined for any sphere S on which f does not vanish. If f and g are two vector fields which can be continuously deformed into one another without vanishing on a sphere S, then they have the same index on S. This index satisfies many nice properties. For instance, there is an analogous formula to that in the theorem earlier in this section which says that, given a smooth vector field X with only isolated critical points on a compact smooth manifold M, the sum of the indices of the X equals the Euler Characteristic of M.

For more information on these topics, we refer to the following books.

1. J. Milnor, *Topology from the differentiable viewpoint*, University Press of Virginia, Charlottesville, Va., 1965

2. M. Hirsch, *Differential Topology*, Springer-Verlag, 1976.