Umlaufsatz

Let Ω be an open connected subset of the plane $\mathbf{R}^2$, and let $\eta = (\eta_1, \eta_2)$ be a C^0 non-vanishing vector field defined in Ω. For $z \in \Omega$,, we wish to define a real number $\zeta_\eta(z)$ which represents the angle between $\eta(z)$ and the positive $x-$direction.

A convenient way to do this uses complex variables. We represent the positive $x-$direction by the complex number 1 (or the real vector $(1, 0)$), and we let $\eta_1(z) = \frac{\eta(z)}{\mid \eta(z) \mid}$ denote the unit vector in the direction of $\eta(z)$. Let $t \in \mathbf{R}$ be any real number such that $e^{it} = \eta_1(z)$. We say that t is *an angle* between $\eta(z)$ and the positive $x-$direction. This is also an angle between $\eta(z)$ and $(1, 0)$. Note that any other real number θ such that $\theta - t = 2\pi n$ for some integer n also gives us an angle between $\eta(z)$ and the positive $x-$direction. Thus, this *angle* really is an element in the circle $\frac{\mathbf{R}}{2\pi\mathbf{Z}}$; i.e., it is well-defined up to an integral multiple of 2π.

Also, we may define $\zeta_\eta(z)$ to be a continuous function of z in a small open ball about z in Ω as follows. The map $\psi(t) = exp(it)$ from $\mathbf{R}$ onto the unit circle $S^1 = \{z \in \mathbf{C} : \mid z \mid^2 = 1\}$ has the property that for each interval U in S^1 of length less than 2π, the inverse image $\psi^{-1}(U)$ is a countable disjoint union of open intervals V_j such that $\psi : V_j \to U$ is a homeomorphism. Pick a small open ball $B_\epsilon(z)$ about z so that for $w \in B_\epsilon(z)$, the vector field $\eta(z)$ lies in a small open interval U in S^1. Then, take any of the open intervals V in $\mathbf{R}$ such that ψ maps V homeomorphically onto U. Let ψ_1 be the inverse map for $\psi \mid V$, and define $\zeta_\eta(z) = \psi_1(\eta(z))$.

Remark. A continuous map $\psi : X \to Y$ between topological spaces X and Y with the property that each point $y \in Y$ has an open neighborhood U so that $\psi^{-1}U$ is a countable disjoint union of open sets in X each of which is mapped homeomorphically by ψ onto U is called a *covering map.* The study of such maps is an important part of the subject of Algebraic Topology. We will not discuss this in detail here, but will only extract the relevant methods.

If $\gamma : [0, 1] \to \Omega$ is a continuous curve in Ω, we may find a continuous function $\zeta_{\eta,\gamma}(t), 0 \leq t \leq 1$ so that $\zeta_{\eta,\gamma}(t)$ is the angle from $\eta(\gamma(t))$ to the positive x-direction as follows. We pick a sequence $0 = t_0 < t_1 < \ldots < t_n = 1$ such that, for $t_j \leq s \leq t_{j+1}$ the vector $\eta(\gamma(s))$ lies in an arc of length less than 2π in S^1. Using the previous construction, we can pick continuous maps $\psi_j : [t_j, t_{j+1}] \to \mathbf{R}$ such that $exp(\psi_j(s)i) = \frac{\zeta(\gamma(s))}{\mid \zeta(\gamma(s)) \mid}$ for all $s \in [t_j, t_{j+1}]$. A given boundary point t_j with $0 < j < n$ may have two values assigned which differ by a multiple of 2π. At t_1 we add a multiple of 2π to ψ_1 so that ψ_0 and ψ_1 agree at t_1. Then do this at each $j = 2, \ldots, n-1$ so that we have a well-defined continuous map on the whole interval $[0, 1]$.

Given a continuous curve $\gamma : [0, 1] \to \Omega$ in Ω and a non-vanishing vector field η in Ω, we define the *angular variation* of η along γ to be

$$j_\eta(\gamma) = \frac{1}{2\pi}(\zeta_{\eta,\gamma}(1) - \zeta_{\eta,\gamma}(0))$$

for any such continuous function $\zeta_{\eta,\gamma}$. This is well-defined (i.e., independent of the choice of continuous angle function $\zeta_{\eta,\gamma}$) for the following reason. Suppose $\zeta_{1,\eta,\gamma}$ and $\zeta_{2,\eta,\gamma}$ were two such functions. Then, $\zeta_{1,\eta,\gamma} - \zeta_{2,\eta,\gamma}$ is continuous and its value is in the set of integral multiples of 2π. Thus, it is constant. So, for some constant c, $\zeta_{1,\eta,\gamma} = \zeta_{2,\eta,\gamma} + c$. This gives

$$\zeta_{1,\eta,\gamma}(1) - \zeta_{1,\eta,\gamma}(0) = \zeta_{2,\eta,\gamma}(1) - \zeta_{2,\eta,\gamma}(0)$$

If η_1 and η_2 are C^1, we can define $\zeta_{\eta,\gamma}$ by the formula

$$j_{\eta,\gamma} = \frac{1}{2\pi}\int_\gamma \frac{\eta_1 d\eta_2 - \eta_2 d\eta_1}{\eta_1^2 + \eta_2^2}$$

In any region in Ω in which η_1 is non-zero, the above line integral is the integral over γ of the 1-form α where $\alpha = \frac{1}{2\pi}d(Arctan(\frac{\eta_2}{\eta_1}))$. Analogously, in a region in which η_2 is non-zero, the line integral is that of the 1-form α with $\alpha = \frac{1}{2\pi}d(ArcCot(\frac{\eta_1}{\eta_2}))$ over γ. Thus, the line integral is the integral of a closed 1-form over γ.

If γ is a Jordan curve and η is a vector field which does not vanish on γ, then $j_\eta(\gamma)$ is called the *index* of η with respect to γ.

Definition. A C^1 positively oriented Jordan curve in $\mathbf{R}^2$ is a C^1 map $\gamma : [a, b] \to \mathbf{R}^2$ from a closed real interval $[a, b]$ such that

1. $\gamma(a) = \gamma(b), \gamma'(a) = \gamma'(b)$
2. $\gamma(t) \neq \gamma(s)$ for $a \leq s < t \leq b$.
3. If $\gamma(t) = (x(t), y(t))$, then $x'(t)^2 + y'(t)^2 \neq 0$ for all $t \in [a, b]$
4. There is an $\epsilon > 0$ such that, for $0 < s < \epsilon$, and any $t \in [a, b]$, we have $(x(t), y(t)) + s(-y'(t), x'(t))$ lies in the bounded region of the complement of the image of γ.

The interpretation of the last condition is that the normal vector to γ at $\gamma(t)$ points into the interior of γ.

Proposition. *Let $\gamma(t) : a \leq t \leq b$ be a Jordan curve in the plane, and let $\xi(t), \eta(t)$ be two continuous vector fields on γ which can be deformed into one another without vanishing. Then, $j_\xi(\gamma) = j_\eta(\gamma)$.*

Proof.

To say that $\xi(t)$ can be deformed into $\eta(t)$ without vanishing means that there is a continuous function $\rho(t, s)$ defined for $a \leq t \leq b, 0 \leq s \leq 1$ such that $\rho(t, 0) = \xi(t), \rho(t, 1) = \eta(t), \forall t$ and $\rho(s, t) \neq 0$ for all (s, t), and $\rho(a, s) = \rho(b, s)$ for all s. For instance, we can use $\rho(t, s) = (1-s)\xi(t)+s\eta(t)$ if $\xi(t)$ and $\eta(t)$ never point in opposite directions on γ.

Let $\phi(s) = j_{\rho(\cdot,s)}(\gamma)$ for fixed s. then, ϕ is a continuous function of s. Since it is integer valued, it must be constant. But, $\phi(1) = j_\eta(\gamma)$ and $\phi(0) = j_\xi(\gamma)$. QED.

Definition. Let γ_1 and γ_2 be two continuous closed curves in $\mathbf{R}^2$, say $\gamma_1 : [0, 1] \to \mathbf{R}^2$, $\gamma_2 : [0, 1] \to \mathbf{R}^2$ are continuous maps with $\gamma_1(0) = \gamma_1(1), \gamma_2(0) = \gamma_2(1)$. We say γ_1 is *homotopic* to γ_2 if there is a continuous function $F : [0, 1] \times [0, 1] \to \mathbf{R}^2$ such that $F(t, 0) = \gamma_1(t)$ and $F(t, 1) = \gamma_2(t)$. When γ_1 is homotopic to γ_2 we also say that γ_1 can be continuously deformed into γ_2.

Definition. A region Ω is *simply connected* if every closed curve in Ω is homotopic to a constant curve.

Thus, the region Ω is simply connected if and only if, for every continuous function $\gamma : [0, 1] \to \Omega$ such that $\gamma(1) = \gamma(0)$, there is a continuous function $F : [0, 1] \times [0, 1] \to \Omega$ such that $F(t, 0) = \gamma(t)$ and $F(t, 1) = \gamma(0) = \gamma(1)$ for all $t \in [0, 1]$.

There is another useful criterion for simply connectivity. A region Ω is simply connected if and only if every continuous function from the unit circle S^1 in $\mathbf{R}^2$ extends to a continous function on the closed unit disk D^2 in R^2.

Proposition. *Let γ_1 and γ_2 be two Jordan curves which can be continuously deformed into one another without passing through a singularity of the vector field f. Then, $j_f(\gamma_1) = j_f(\gamma_2)$.*

The proof is similar to that of the previous proposition.

Definition. Let x_0 be an isolated critical point of a C^1 vector field f in the plane. Let γ be a small C^1 positively oriented Jordan curve whose interior contains x_0. The index $j_f(\gamma)$ of f with respect to the curve γ is called the *index* of the critical point x_0 (with respect to the vector field f. It is denoted $Ind(f, x_0)$ or $j_f(x_0)$.

Note that if γ_1, γ_2 are two positively oriented C^1 curves whose interiors contain x_0 and γ_1 can be continuously deformed into γ_2 without passing through a critical point of f, then $j_f(\gamma_1) = j_f(\gamma_2)$.

Hence the index is independent of the small positively oriented Jordan curve chosen to calculate it.

Examples. Sources and sinks have index +1, saddles have index -1.

Lemma. *Let f be a C^1 vector field which does not vanish on the closure of the interior of a Jordan curve γ. Then, $j_f(\gamma) = 0$.*

Proof. Let A be the interior of γ (i.e., the bounded component of $\mathbf{R}^2 \setminus \gamma$). The set A is simply connected. So the curve can be continously deformed to a very small Jordan curve γ_1 in A. But, since f does not vanish in A, the index $j_f(\gamma_1)$ is zero if γ_1 is small enough. QED

Theorem.(Umlaufsatz) *Let γ be a C^1 positvely oriented Jordan curve in the plane and let γ' be its tangent vector field. Then,*

$$j_{\gamma'}(\gamma) = 1$$

Proof.

The result is clearly independent of the position of the curve γ in the plane. So, translate the curve γ so that it is above and tangent to the $x-$axis. Let the curve be given by $s \to \gamma(s) = (x(s), y(s))$ with $0 \leq s \leq 1$, $\gamma(0) = \gamma(1)$ and $\gamma(s) \neq \gamma(t)$ for $s < t < 1$.

Consider the triangle $\Delta = \{(s,t) : 0 \leq s \leq t \leq 1\}$, and the subset $\Delta_0 = \{(s,t) : 0 \leq s < t \leq 1\}$.

Let $\eta(s,t) = \frac{\gamma(t) - \gamma(s)}{t-s}$.

This function is continuous for $t \neq s$, and extends to a continuous function on the closed triangle Δ which agrees with $\gamma'(t)$ for $s = t$. Moreover, since Δ is simply connected, there is a continuous function $\zeta(s,t)$ defined on Δ so that $\zeta(s,t)$ is the angle from $\eta(s,t)$ to the positive $x-$direction.

It is clear that $j_{\gamma'}(\gamma) = \frac{1}{2\pi}(\zeta(1,1) - \zeta(0,0))$.

Now, $\zeta(1,1) - \zeta(0,0) = \zeta(1,1) - \zeta(0,1) + \zeta(0,1) - \zeta(0,0)$. Considering $\zeta(0,t)$ as t varies from 0 to 1 we see that $\zeta(0,1) - \zeta(0,0) = \pi$ since $\eta(0,t)$ always points into the upper half-plane. Similarly, $\eta(s,1)$ always points into the lower half-plane, so, as s varies from 0 to 1, we see that $\zeta(1,1) - \zeta(0,1) = \pi$. QED

Proposition. *Let γ be a non-trivial periodic orbit of a C^1 planar vector field. Then, γ is a Jordan curve. Let A be its interior. Then, f has a critical* *point in A.*

Proof. By the Umlaufsatz, $j_f(\gamma) = \pm 1$ depending on whether γ is positively or negatively oriented as a solution of the vector field f. (Strictly speaking, if γ is given some parametrization so that it is positively oriented, then with respect to that parametrization, $j_f(\gamma) = 1$. This is true whether the parametrization as a solution makes γ positively or negatively oriented). If f had no critical points in A, the previous Lemma would give $j_f(\gamma) = 0$ which is a contradiction. QED

Proposition. *Let f be a C^1 vector field with only finitely many critical points $x_1, x_2, \ldots, x_n$ in the interior of a positively oriented Jordan curve γ. Then,*

$$j_f(\gamma) = Ind(f, x_1) + \ldots + Ind(f, x_n)$$

Proof. Consider small positively oriented Jordan curves γ_i about x_i in the interior of γ. Join γ to each γ_i by an arc η_i so that the $\eta_i's$ are disjoint. We may split the curves η_i into small arcs going in opposite directions η_{i1}, η_{i2} and use pieces of γ, γ_i with these new curves to get a simple closed positively oriented curve $\tilde{\gamma}$ whose interior contains no critical points. Thus, $j_f(\tilde{\gamma}) = 0$.

But $j_f(\tilde{\gamma})$ is approximately

$$j_f(\gamma) - \sum Ind(f, x_i).$$

Passing to the limit as the curves η_{ij} approach $\pm\eta_i$, proves the result. QED.

Definition. The standard $n-simplex$ is the set $\Delta_n = \{x \in \mathbf{R}^{n+1} : x = (x_0, \ldots, x_n), x_i \geq 0 \ \forall i, \sum_i x_i = 1\}$. A topological $n-simplex$ in $\mathbf{R}^p$ is the homeomorphic image of Δ_n (or a homeomorphism σ from Δ_n into $\mathbf{R}^p$).

Thus, a 0-simplex is a point, a 1-simplex is a homeomorphically embedded line segment, a 2-simplex is a homeomorphically embedded triangle, etc.

Definition. Suppose Δ_n is the standard $n-$simplex. Its *interior* is the set $\{x \in \Delta_n : x_i > 0 \ \forall i\}$. For $1 \leq k \leq n+1$, let $\mathcal{A}_k$ be the set of $k-$tuples $i_1 < i_2 < \ldots < i_k$ of distinct integers in $0, \ldots, n$. The $(k-1)-$face in Δ_n determined by a $k-$tuple in $\mathcal{A}_k$ is the set of points $x = (x_0, x_1, \ldots x_n) \in \Delta_n$ such that $\sum_{1 \leq j \leq k} x_{i_j} = 1$. A 0-face is called a *vertex* and a 1-face is called an *edge*. An open $k-$face is a $k-face$ minus all of its $(k-1)-$subfaces.

Thus, a 0-face is one of the $e_i's$, an edge is the line segment joining a pair of distinct vertices, etc. Note that there is an affine embedding from $\mathbf{R}^{k+1}$ to

$\mathbf{R}^{n+1}$ (linear embedding plus translation) carrying the standard $k-$simplex onto any $k-$face of Δ_n.

If $\sigma : \Delta_n \to S$ is a representation of the topological $n-$simplex S, then a $k - face$ of S is the image by σ of a $k-$face of Δ_n. *Vertices* of S are images of vertices of Δ_n, *edges* of S are images of edges of Δ_n, etc.

A triangulation of a subset K of $\mathbf{R}^p$ is a collection of topological simplexes $\mathcal{T}$ such that

1. $\bigcup_{\sigma \in \mathcal{T}} \sigma = K$
2. If $\sigma \in \mathcal{T}$ and τ is a face of σ, then $\tau \in \mathcal{T}$.
3. If $\sigma \in \mathcal{T}$ and $\tau \in \mathcal{T}$, then $\sigma \bigcap \tau$ is a common face of both σ and τ.

The *dimension* of an $n-$simplex is n. A *triangulatable* set is a set which has some triangulation. If K can be triangulated by finitely many simplexes, and the largest dimension of one of those simplexes is n, we call K an $n-$complex.

We will be interested in 2-complexes. Then, we call the 2-faces simply faces, and we only have vertices, edges and faces among the simplexes involved.

Theorem. *Let K be an n-complex. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two finite triangulations of K. For $i = 1, 2, 0 \le j \le n$, let b_{ij} be the number of $j-$simplexes in $\mathcal{T}_i$. Then,*

$$\chi(\mathcal{T}_1) \equiv \sum_{j=0}^{n} (-1)^j b_{1j} = \sum_{j=0}^{n} (-1)^j b_{2j} \equiv \chi(\mathcal{T}_2)$$

The number $\chi(\mathcal{T}_j)$ is called the *Euler characteristic* of the triangulation. From the theorem one can define the Euler characteristic of a finite complex using the Euler characteristic of any of its triangulations.

This theorem will not be proved here. We only mention that a proof can be given using the concept of homology. With this concept one defines another number and shows that the Euler characteristic of any triangulation equals this number, so any two must be equal.

Theorem. *Suppose Ω is a bounded region in the plane bounded by finitely many positively oriented Jordan curves $\gamma_1, \ldots, \gamma_n$. (such a region is called a multiply connected domain). Let $\bar{\Omega} = \Omega \bigcup_i \gamma_i$ be the closure of Ω. Let f be a C^1 vector field such that each boundary curve γ_i is a periodic solution of f*

and the parametrizations by solutions make γ_i positively oriented. Suppose in addition that f has only finitely many critical points $x_1, \ldots, x_k$ in Ω.

Then,

$$\sum_i Ind(f, x_i) = \chi(\bar{\Omega})$$

Proof. Using the standard little cuts joining boundary curves, we see that the sum of the indices of f at the critical points equals 2 - (number of boundary curves). But this last number is the Euler characteristic of $\bar{\Omega}$.

Here is an alternate proof. There is a single curve among the $\gamma_i's$ such that all the others are in the interior region of this curve. Call this curve γ_1.

Construct a new vector field $\tilde{f}$ on the closure of the interior of γ_1 (the outer curve) which equals f in the closure of the region Ω and adds a single critical point p_i of index +1 in the interior of each $\gamma_i, i > 1$.

Then, $\tilde{f}$ has the critical points $x_i, i \geq 1, p_j, j > 1$ inside γ_1. By a previous theorem,

$$\sum_i Ind(\tilde{f}, x_i) + \sum_j I(\tilde{f}, p_j) = j_{\tilde{f}}(\gamma_1) = 1$$

Hence,

$$\begin{aligned} \sum_i Ind(f, x_i) &= 1 - (\text{number of internal boundary curves}) \\ &= 2 - (\text{number of boundary curves}) \end{aligned}$$

QED