Degree for maps of the circle

Let $S^1 = \{z \in \mathbf{C} : |\, z \,| = 1\}$.

Let $i = \sqrt{-1}$.

The *standard covering projection* from $\mathbf{R}$ onto S^1 is the map $\rho : \mathbf{R} \to S^1$ defined by

$$\rho(t) = exp(2\pi it)$$

Definition. An *open interval* in S^1 is the homeomorphic image by ρ of a real open interval. Any such interval clearly has length less than 1.

Also, note the if U is an open interval in S^1, then $\rho^{-1}(U)$ is a countable disjoint union of open intervals $\tilde{U}_j$ such that $\rho \mid \tilde{U}_j$ is a homeomorphism onto U for each j. One says that U is *evenly covered* by ρ.

Let $I = [0, 1]$ be the closed unit real interval.

Let X be a topological space (it will be an open connected set in the plane in our application). A *path* in X is a continuous map $\gamma : I \to X$. If $p = \gamma(0)$ and $q = \gamma(1)$, we say that γ is a path *from* p to q.

Definition.

1. The topological space X is *path connected* if for any two points p, q in X, there is a path from p to q (in X).

2. The topological space X is *locally path connected* if it is path connected, and, for any $x \in X$ and any neighborhood U of x, there is a path connected neighborhood V of x such that $x \in V \subset U$.

Exercises

1. A path connected topological space is connected.

2. A connected, locally path connected topological space is path connected.

3. Give examples of subsets X, Y of the plane $\mathbf{R}^2$ such that X is connected, but not path connected, and Y is path connected, but not locally path connected.

All spaces we consider in the rest of this section are assumed to be path connected and locally path connected.

Let X, Y be topological spaces, let $A \subset X$, and let γ_0, γ_1 be two continuous maps from $X \to Y$ such that γ_0 and γ_0 agree on A. That is, $\gamma_0(x) = \gamma_1(x)$ for $x \in A$.

Let $X \times I$ denote the product space, and denote $X_0 = X \times \{0\}$, $X_1 = X \times \{1\}$.

Each of the maps γ_i is completely determined by associated map $\bar{\gamma}_i : X \times \{i\} \to Y$ where $\bar{\gamma}_i((x,i)) = \gamma_i(x)$ for $i = 0, 1$. Thus, we will abuse the notation slightly and write the maps $\bar{\gamma}_i$ also as γ_i, letting the context make the difference clear. Sometimes we use the words, *we identify* γ_i with $\bar{\gamma}_i$ for $i = 0, 1$.

We say that γ_0 is *homotopic* to γ_1 *relative* to A if there is a continuous map $F : X \times I \to Y$ such that

1. $F \mid X_0 = \gamma_0$ and $F \mid X_1 = \gamma_1$
2. $F(x,t) = \gamma_0(x) = \gamma_1(x)$ for all $x \in A$.

This makes precise the intuitive notion of saying that γ_0 can be *continuously deformed* into γ_1 by a family $F_s(\cdot) = F(\cdot, s)$ of continuous maps from X to Y with each map F_s not changing at points of A.

We will use the notation $\gamma_0 \simeq_A \gamma_1$ when γ_1 is homotopic to γ_2 relative to A. When A is the empty set, the second condition above is always true (there are no points in $\emptyset$ to make it false), so it does not have to be stated. In that case, we simply say that γ_0 is *homotopic* to γ_1.

Now, let us specialize this notion to the case of $X = I$ itself and $A = \partial I = \{0, 1\}$. We also replace the image space Y by X.

Thus, let $\gamma_0 : I \to X$ be two paths in the path connected, locally path connected topological space X.

Then $\gamma_0 \simeq_\partial \gamma_1$ means that $\gamma_0(0) = \gamma_1(0)$, $\gamma_0(1) = \gamma_1(1)$, and there is a continuous map $F : I \times I \to X$ such that

1. $F(t, 0) = \gamma_0(t)$ and $F(t, 1) = \gamma_1(t)$ for all $t \in I$, and
2. $F(0, s) = \gamma_0(0) = \gamma_1(0)$ and $F(1, s) = \gamma_0(1) = \gamma_1(1)$ for all $s \in I$.

Thus, F provides a family $F_s(\cdot) = F(\cdot, s)$ of paths from γ_1 to γ_2 keeping the endpoints unchanged as s varies.

As above, the map F is called a *homotopy from γ_1 to γ_2 relative ∂I.*

We write $\gamma_0 \simeq_\partial \gamma_1$.

Given two paths γ, η with $\gamma(1) = \eta(0)$, one defines the *concatenation path*

$$\gamma\sharp\eta = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \eta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Whenever we write $\gamma\sharp\eta$, we assume that $\gamma(1) = \eta(0)$.

Definition.

1. Let $p \in X$. A *loop* in X based at p is a path $\gamma : I \to I$ such that $\gamma(0) = \gamma(1) = p$. Note that we do not require that $\gamma \mid [0,1)$ be injective.

2. A loop $\gamma : I \to X$ based at p is *null-homotopic* if it is homotopic relative to ∂I to the constant loop $\zeta(t) \equiv p$ for all $t \in I$.

If $\gamma : I \to X$ is a loop based at p, then we write $\gamma \simeq_\partial 0$ to mean that γ is null-homotopic. Also, we use the term *homotopic to zero* for this concept.

Given a path γ in X, the *reverse or inverse* path is the path $-\gamma(t) = \gamma(1-t)$.

Exercise

1. If $\gamma_0 \simeq_\partial \gamma_1$ and $\eta_0 \simeq_\partial \eta_1$, then $\gamma_0\sharp\eta_0 \simeq_\partial \gamma_1\sharp\eta_1$

2. For any γ, we have $\gamma\sharp(-\gamma) \simeq_\partial 0$

Let $\Omega(X,p)$ denote the collection of all loops based at p.

The relation $\gamma_0 \simeq_\partial \gamma_1$ is an equivalence relation on $\Gamma(X,p)$. The quotient set

$$\Omega(X,p)/\simeq_\partial$$

is denoted $\pi_1(X,p)$.

The operation $\sharp$ on $\Omega(X,p)$ pushes down to an operation on $\pi_1(X,p)$ which turns $\pi_1(X,p)$ into a group. It is called the *fundamental group* of (X,p) or just X.

If γ represents an element of $\pi_1(X,p)$, then $-\gamma$ represents its inverse for the operation induced by concatenation.

It is a fact that if p, q are two points in X, then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.

Indeed, an isomorphism can be constructed as follows. Let η be a path from p to q. For $\gamma \in \Omega(X,p)$, let $\Phi(\gamma) = (-\eta\sharp\gamma)\sharp\eta$. This is an element of

$\Omega(X, q)$. It can be shown that if $\gamma_1 \simeq_\partial \gamma$, then $\Phi(\gamma_1) \simeq_\partial \Phi(\gamma)$. Thus, we get a well-defined map, $\tilde{\Phi} : \pi_1(X, p) \to \pi_1(X, q)$. We leave it as an exercise to show that $\tilde{\Phi}$ is a group isomorphism.

A topological space X is called *simply connected* if for some $p \in X$, every loop γ based at p is null-homotopic. That is, the group $\pi_1(X, p)$ is the trivial group consisting of a single element. Since $\pi_1(X, p)$ is isomorphic to $\pi_2(X, q)$ for any $p, q \in X$, we can use any $p \in X$ to test for simple-connectivity.

Definition Let X be a path connected, locally connected space, and let $f : X \to S^1$ be continuous. A *lift* of f (or a *lift to* $\mathbf{R}$) is a continuous map $\tilde{f} : X \to \mathbf{R}$ such that $\rho\tilde{f} = f$.

Proposition 0.1 *(Path lifting property) Let $\gamma : I \to S^1$ be a path in S^1. Then, there is a lift $\tilde{\gamma}$ of γ. If $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are any two such lifts, then there is an integer n such that $\tilde{\gamma}_1(t) = \tilde{\gamma}_2(t) + n$ for all $t \in I$.*

Proof.

Let us first construct a lift $\tilde{\gamma}$.

Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a collection of open intervals in S^1 whose union covers S^1 such that, for each $i \in [1, n]$,

$$\rho^{-1}(U_i) \text{ is a disjoint union } \{V_{ij}\} \text{ of open intervals in } \mathbf{R} \text{ and } diam(V_{ij}) < 1/4 \tag{1}$$

and

$$\text{if } j \neq k, \text{ then } dist(V_{ij}, V_{ik}) > 1/2. \tag{2}$$

Let $t_0 = 0 < t_1 < \ldots < t_k = 1$ be a finite collection of points in I so that the image $\gamma([t_i, t_{i+1}])$ is contained in one of the intervals U_j for each $0 \leq i < k$.

We construct a lift $\tilde{\gamma}_i$ of $\gamma \mid [t_0, t_i]$ for each i. At the end of the induction, we simply set $\tilde{\gamma} = \tilde{\gamma}_k$.

First, pick a real number θ_0 so that $\rho(\theta_0) = \gamma(t_0)$, and let U_{i_0} be one of the intervals in $\mathcal{U}$ which contains $\gamma([t_0, t_1])$.

Let V_{i_0, j_0} be the real interval in $\rho^{-1}(U_{i_0})$ which contains θ_0.

The map ρ maps V_{i_0, j_0} homeomorphically onto U_{i_0}. Let h_0 be its inverse map.

Let $\tilde{\gamma}_0 = h_0 \circ \gamma \mid [t_0, t_1]$. This is the first step of our induction.

Now, for $0 \leq s < k-1$, assume that $\tilde{\gamma}_s$ has been defined as a lift of $\gamma \mid [t_0, t_{s+1}]$.

We wish to define $\tilde{\gamma}_{s+1}$ from $[t_0, t_{s+2}]$ to $\mathbf{R}$ as a lift of $\gamma \mid [t_0, t_{s+2}]$.

Let $\theta_{s+1} = \tilde{\gamma}_s(t_{s+1})$, and let $U_{i_{s+1}}$ be the interval in $\mathcal{U}$ containing $\rho(\theta_{s+1})$, let $V_{i_{s+1}, j_{s+1}}$ be the interval in $\rho^{-1}(U_{i_{s+1}})$ containing θ_{s+1}.

Let h_{s+1} be the inverse of ρ restricted to $U_{i_{s+1}}$.

Define $\tilde{\gamma}_{s+1}$ to be $\tilde{\gamma}_s$ on $[t_0, t_{s+1}]$ and $h_{s+1} \circ \gamma$ on $[t_{s+1}, t_{s+2}]$. Then, $\tilde{\gamma}_{s+1}$ provides the lift on $[t_0, t_{s+2}]$. By induction, we can go all the way up to $[t_0, t_k] = I$.

Now, suppose that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two lifts which agree at 0.

Let

$$A = \{t \in I : \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)\}$$

Then, A is non-empty. We leave it as an exercise using (2) that A is both open and closed. Since I is connected, we must have $A = I$.

Thus if the two lifts agree at a single point, they must agree everywhere.

Now, consider two arbitrary lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$. Since $\rho(\tilde{\gamma}_1(0)) = \rho(\tilde{\gamma}_2(0))$, there is an integer n such that $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) + n$.

The map $t \to \tilde{\gamma}_2(t) + n$ is a lift of γ which agrees with $\tilde{\gamma}_1$ at 0. Hence, they agree at every t. QED.

Proposition 0.2 *(Homotopy lifting property) Let $\gamma_i : I \to S^1$, $i = 1, 2$ be two paths in S_1 which are homotopic via the homotopy F. Then, there is a lift $\tilde{F}$ of F to $\mathbf{R}$ such that $\tilde{F}$ is a homotopy from $\tilde{F} \mid I \times \{0\}$ to $\tilde{F} \mid I \times \{1\}$.*

Moreover, any two such lifts $\tilde{F}_1, \tilde{F}_2$ differ by an integer translation.

The proof is similar to that of Proposition (0.1).

We cover S^1 with a finite collection of open intervals $\mathcal{U} = \{U_{i_j}\}$ satisfying (1) and (2).

Next, we pick finite sequences of points $(t_i, u_j) \in I \times I$ with $0 \leq i \leq N$, $0 \leq u \leq N$ for some large enough N such that the $F-$image of each box $B_{ij} = \{(t, u) : t_i \leq t \leq t_{i+1}, u_j \leq u \leq u_{j+1}\}$ is contained in a single element of $\mathcal{U}$. We then construct the lift $\tilde{F}$ by inductively constructing along the boxes B_{ij} along each level $i = \mathit{const}$, letting j go from 0 to N, and then letting i go from 0 to N. Since the image of each closed box B_{ij} is an open interval U_{ij} and the connected components of $\rho^{-1}(U_{ij})$ are not too close relative to their diameters, the induction can continue.

The uniqueness up to integer translations is proved as in Proposition (0.1) as well. QED.

Remark. If the homotopy F is rel ∂, then the left $\tilde{F}$ can also be chosen to be rel ∂.

Proposition 0.3 *Let γ be a path in $\mathbf{R}$ such that $\gamma(1)-\gamma(0)$ is an integer n, and let $\eta(t)=(1-t)\gamma(0)+t\gamma(1)$ be the affine path parametrizing the interval from $\gamma(0)$ to $\gamma(1)$. Then, the curves γ and η are homotopic relative ∂I.*

Proof. The map $F_s(t)=(1-s)\gamma(t)+s\eta(t)$ gives the required homotopy. QED.

Thus, any path γ in $\mathbf{R}$ is homotopic relative to ∂I to an affine path.

Lemma 0.4 *Let $\gamma : I \to S^1$ be any loop in S^1. Then, $\gamma \simeq_{\partial} 0$ if and only if any lift $\tilde{\gamma} : I \to \mathbf{R}$ to $\mathbf{R}$ is a loop in $\mathbf{R}$.*

Proof. First observe that if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two lifts of γ, then $\gamma_1(0)=\gamma_2(0)$ and $\gamma_1(1)=\gamma_2(1)$. Thus, γ_1 is a loop iff γ_2 is also a loop.

Next, assume that some lift $\tilde{\gamma}$ is a loop in $\mathbf{R}$. The point $p=\tilde{\gamma}(0)=\tilde{\gamma}(1)$ in $\mathbf{R}$ is such that $\rho(\tilde{\gamma}(1))=\rho(\tilde{\gamma}(0))$ in S^1. But, $\rho\circ\tilde{\gamma}=\gamma$, so γ is indeed a loop in S^1.

For the converse, assume that $\gamma \simeq_{\partial} 0$. Let $F : I\times I \to S^1$ be a homotopy (relative to ∂I) such that $F \mid I\times\{0\}=\gamma$ and $F \mid I\times\{1\}=c$ (i.e. $F(t,1)=c$ for all $t\in I$) where $\rho(c)=\gamma(0)$.

Using the homotopy lifting property, we lift F to a homotopy $\tilde{F} : I\times I \to \mathbf{R}$ relative to ∂I so that $\rho\circ\tilde{F}=F$.

The curve $\tilde{\gamma}=\tilde{F}\mid I\times\{0\}$ is a lift of γ. We wish to show that $\tilde{\gamma}(1)=\tilde{\gamma}(0)$.

But, by the property that $\tilde{F}$ is a relative homotopy, we have that $\tilde{F}\mid\{0\}\times I$ and $\tilde{F}\mid\{1\}\times I$ are constant functions. That is, $\tilde{F}$ is constant on the left and right boundaries of $I\times I$. Since $\tilde{F}\mid I\times\{1\}=c$, $\tilde{F}$ is also constant on the upper boundary of $I\times I$. Since $\tilde{F}$ is continuous, (and the union of the left, right, and upper boundaries of $I\times I$ is connected), these three constants must be equal. This implies that $\tilde{\gamma}(1)=\tilde{\gamma}(0)$, so $\tilde{\gamma}$ is a loop. QED.

Proposition 0.5 *Let X be simply connected and let $f : X\to S^1$ be a continuous map. Then, there is a lift $\tilde{f} : X\to\mathbf{R}$ of f to $\mathbf{R}$. Any two such lifts differ by an integer translation.*

Proof. Pick a point $p \in X$, and let $\theta_0 \in \mathbf{R}$ be a point in $\rho^{-1}(f(p))$. Let $q \in X$, and let $\gamma : I \to X$ be a path from p to q. The map $f \circ \gamma : I \to S^1$ is then a path in S^1. Lift this path to a path $\tilde{\gamma} : I \to S^1$.

Define

$$\tilde{f}(q) = \tilde{\gamma}(1). \tag{3}$$

We claim:

($\star$) The definition (3) is independent of the path γ from p to q and the choice of any lift of γ. That is, if $\gamma_1,\ \gamma_2$ are two paths in X from p to q, and $\tilde{\gamma}_1,\ \tilde{\gamma}_2$ are any lifts of $f \circ \gamma_1,\ f \circ \gamma_2$, respectively, such that $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$, then we have $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$

Assume the claim for the moment.

Then, we get a well-defined lift $\tilde{f}$ of f defined by (3).

We wish to show that it is continuous at each point $q \in X$.

Fix a path γ from p to q in X with lift $\tilde{\gamma}$. Then, $\tilde{f}(q) = \tilde{\gamma}(1)$.

Let V be any open set in $\mathbf{R}$ containing $\tilde{f}(q)$. We wish to find an open set U in X containing q such that $\tilde{f}(U) \subset V$.

Let V_0 be a small interval in V about $\tilde{f}(q)$ so that $\rho \mid V_0$ maps V_0 homeomorphically onto its image V_0' in S^1.

Continuity of f gives us a neighborhood U_0 of q in X so that $f(U_0) \subset V_0'$. Since X is locally path connected, we may assume that U_0 is path connected.

For $q_1 \in U_0$, let $\gamma_1 : I \to X$ be a path in U_0 from q to q_1. Then, the path $\eta \overset{\text{def}}{=} \gamma \sharp \gamma_1$ is a path in X from p to q_1 and $\eta(1) = f(q_1)$. The lift $\tilde{\eta}$ of η to $\mathbf{R}$ satisfies $\tilde{f}(q_1) = \tilde{\eta}(1) = (\rho \mid V_0)^{-1}(\eta(1))$. But, this last point lies in V_0. Hence $\tilde{f}(q_1) \in V_0 \subset V$ as required.

It remains to prove ($\star$).

Let γ_1, γ_2 be two paths from p to q with lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ of $f \circ \gamma_1,\ f \circ \gamma_2$, respectively.

The curve $\eta \overset{\text{def}}{=} (-\gamma_2)\sharp\gamma_1$ is then a loop in X based at q and $\eta(0) = \gamma_1(1) = \gamma_2(1)$. Since X is simply connected, we have $\eta \simeq_\partial 0$ in X.

This implies that $\eta_1 \overset{\text{def}}{=} f \circ \eta \simeq_\partial 0$ in S^1. Hence, by Lemma 0.4, the lift $\tilde{\eta}_1$ of this last loop is a loop in $\mathbf{R}$. Thus, $\tilde{\eta}_1(1) = \tilde{\eta}_1(0)$. But, $\tilde{\eta}(1) = \tilde{\gamma}(1)$ and $\tilde{\eta}(0) = \tilde{\gamma}_2(1)$. QED.

Now consider a continuous map $f : S^1 \to S^1$.

Since $\mathbf{R}$ is simply connected, the map $f \circ \rho\mathbf{R} \to S^1$ has a lift $\tilde{f} : \mathbf{R} \to \mathbf{R}$ which satisfies $\rho\tilde{f} = f\rho$. We call $\tilde{f}$ a *lift of to* $\mathbf{R}$ *which covers* f.

Any two such lifts differ by an integer translation. We define the *degree* of f to be $\tilde{f}(1) = \tilde{f}(0)$ for any lift which covers f. This is an integer. We denote it by $deg(f)$.

The next theorem is fundamential.

Theorem 0.6 *Let* $f : S^1 \to S^1$ *and* $g : S^1 \to S^1$ *be two continous self-maps of the circle. Then,* f *is homotopic to* g *iff deg(f) = deg(g).*

Proof

Let $\bar{1}$ denote the complex number $1 + 0 \cdot \sqrt{-1}$.

We first reduce to the case in which $f(\bar{1}) = \bar{1}$. That is, f has $\bar{1}$ as a fixed point.

Let $\theta \in \mathbf{R}$ and let $T_\theta(x) = x + \theta$ be the translation on $\mathbf{R}$ by θ. Let $R_\theta : S^1 \to S^1$ be the rotation by angle $2Pi\theta$ given by $R_\theta(z) = e^{2\pi\theta}z$.

It is easy to see that T_θ is a lift of R_θ with $T_\theta(0) = \theta$. We call T_θ the *standard lift* of $\mathbf{R}_\theta$.

Given $f : S^1 \to S^1$, let θ be such that $R_\theta(\theta) \circ f(\bar{1}) = \bar{1}$.

We leave it as an exercise that any rotation R_θ on S^1 is homotopic to the identity map id. It follow that $R_\theta \circ f \simeq f$.

Let $\tilde{f}$ be a lift of f to $\mathbf{R}$. Then, $T_\theta \circ \tilde{f}$ is a lift of $R_\theta \circ f$.

Since $\mathbf{T}_\theta \circ \tilde{f}(x) = \tilde{f}(x) + \theta$, it is clear that $T_\theta \circ \tilde{f}(1) - T_\theta \circ \tilde{f}(0) = \tilde{f}(1) - \tilde{f}(0)$, so

$$deg(f) = deg(R_\theta \circ f).$$

Similarly, let θ_1 be such that $R_{\theta_1} \circ g(\bar{1}) = \bar{1}$. Then,

$$deg(f) = deg(R_{\theta_1} \circ f),$$

and $f \simeq g$ iff $R_\theta \circ f \simeq R_{\theta_1} \circ g$

$$deg(g) = deg(R_{\theta_1} \circ g).$$

It follows that we may replace f by $R_\theta \circ f$ g by $R_{\theta_1} \circ g)$, so it suffices to prove the theorem assuming f and g both fix $\bar{1}$.

Consider the maps $f_1 \stackrel{\text{def}}{=} f \circ \rho \mid [0,1]$ and $g_1 \stackrel{\text{def}}{=} g \circ \rho \mid [0,1]$. These are both paths in S^1, and $f \simeq g$ iff $f_1 \simeq_\partial g_1$.

Assume first that $f \simeq g$. Then $f_1 \simeq_\partial g_1$, and $f_1 \sharp (-g_1) \simeq_\partial 0$. It follows from Lemma 0.4 that any lift of $f_1 \sharp (-g_1)$ is a loop in $\mathbf{R}$.

Choose lifts $\tilde{f}_1$ of f_1 and $\tilde{g}_1$ of g_1, respectively so that $\tilde{f}_1(1) = 0 = \tilde{g}_1(0)$. Then, $\tilde{f}_1 \sharp (-\tilde{g}_1)$ is defined, and it is a lift of $f_1 \sharp (-g_1)$. Since the latter curve is a null-homotopic loop in S^1, the curve $\tilde{f}_1 \sharp (-\tilde{g}_1)$ is a loop in $\mathbf{R}$. This implies that $\tilde{f}_1(1) - \tilde{f}_1(0) = \tilde{g}_1(1) - \tilde{g}_1(0)$. So, $deg(f_1) = deg(g_1)$.

Conversely, suppose that $deg(f_1) = deg(g_1)$. Let $\tilde{f}_1, \tilde{g}_1$ be lifts of f_1, g_1, respectively such that $\tilde{f}_1(0) = 0$ and $\tilde{g}_1(0) = 0$. Since $deg(f) = \tilde{f}_1(1) - \tilde{f}_1(0) = deg(g) = \tilde{g}_1(1) - \tilde{g}_1(0)$, we have that $\tilde{f}_1(1) = \tilde{g}_1(1)$. Letting n denote this common value, Propostion 0.3, gives that $\tilde{f}_1 \simeq_\partial [0,n]$ and $\tilde{g}_1 \simeq_\partial [0,n]$. It follows that $\tilde{f}_1 \simeq_\partial \tilde{g}_1$, and, hence $f_1 = \rho \circ \tilde{f}_1 \simeq_\partial \rho \circ \tilde{g}_1 = g_1$. QED.