1. Construct a C^∞ vector field X in the plane with exactly eight critical points such that half of them are hyperbolic saddle points and half of them are hyperbolic sinks.

2. Let $\mathcal{X}^1(\mathbf{R}^n)$ be the set of C^1 vector fields defined on all of $\mathbf{R}^n$. For $X \in \mathcal{X}^1(\mathbf{R}^n)$, define

$$| X |_1 = \sup_{x \in \mathbf{R}^n} \max(| X(x) |, | DX(x) |)$$

   This satisfies the usual properties of a norm except that it may be infinite.

   Let $r \geq 0$.

   We say that two vector fields X and Y on $\mathbf{R}^n$ are *topologically equivalent* if there is a homeomorphism h from $\mathbf{R}^n$ onto itself carrying orbits of X to orbits of Y.

   If X and Y are complete vector fields (i.e., solutions defined for all time), with flows ϕ, η, respectively, we say that X and Y are topologically conjugate if there is a homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ such that, for each $t \in \mathbf{R}$, we have

$$h \phi_t h^{-1} = \eta_t$$

   We also use the terms C^0-equivalent or C^0-conjugate for topological equivalent or topologically conjugate, respectively. If the maps h can be taken to be C^r with $r \geq 1$, then we use the terms C^r-equivalent or C^r-conjugate, respectively. The maps h are then called a topological (C^r) equivalence or topological (C^r) conjugacy.

   We say that X is *structurally stable* if there is an $\epsilon > 0$ such that if $Y \in \mathcal{X}^1(\mathbf{R}^n)$ and $| Y - X |_1 < \epsilon$, then Y is topologically equivalent to X.

3. Prove that if X is structurally stable, then there is an $\epsilon > 0$ such that if $| Y - X |_1 < \epsilon$, then Y is also structurally stable.

4. Suppose that X is structurally stable. Prove that every isolated critical point of X is hyperbolic.

   Remark. This is actually true without the assumption that the critical point in isolated.

5. Give an example of two C^∞ planar vector fields X and Y which are topologically equivalent, but not topologically conjugate.

6. Let X be a complete C^∞ vector field in $\mathbf{R}^n$, and let ρ be a C^∞ diffeomorphism from $\mathbf{R}^n$ onto itself. Prove that the push-forward vector field $\rho_\star(X)$ is C^∞ conjugate to X.

7. Sketch the solutions of each of the following differential equations, indicating whether the critical points are sources, sinks, saddles, or centers.

   (a) $$\begin{aligned} x' &= y \\ y' &= x(x-1)(x-2) \end{aligned}$$

   (b) $$\begin{aligned} x' &= 3y \\ y' &= -x(x-1)(x-2)(x-3)(x-4)-y \end{aligned}$$

   (c) $$\begin{aligned} x' &= 3y \\ y' &= -(x^3-x)(x-3)+2y \end{aligned}$$