SUMMARY OF MATERIAL ON THE INDEX.

The following is a summary of the material on the notion of index for non-zero vector fields along a curve and for isolated critical points.

1 Paths, Lifts, and Homotopy

You should know the definitions and statements of the results in this section. You do not have to know the proofs.

Let X be a subset of R^n for some n, and let $I = [0, 1]$ be the closed unit interval in the reals.

Definitions.

1. A path in X is a continuous map $\gamma : [0, 1] \to X$.
2. Given the path $\gamma : [0, 1] \to X$, define its *negative* path to be $-\gamma : [0, 1] \to X$ by

$$-\gamma(t) = \gamma(1 - t)$$

   Sometimes, $-\gamma$ is called the *oppositie* or *inverse* path of γ and is written γ^{-1}.
3. Let $\gamma_1 : [0, 1] \to X, \gamma_2 : [0, 1] \to X$ be two paths in X such that $\gamma_1(1) = \gamma_2(0)$. Define the *concatenation* of γ_1 and γ_2 to be the path $\gamma_1 \sharp \gamma_2$ defined by

$$\gamma_1 \sharp \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

4. The space X is *path connected* if for any two points $p, q \in X$, there is a path γ such that $\gamma(0) = p$ and $\gamma(1) = q$. Such a path is called a *path from p to q*.

5. The space X is *connected* if it cannot be written as a disjoint union of two non-empty open subsets.

6. The space X is *locally path connected* if given any $p \in X$ and any open set U containing p, there is a path connected open set V such that $p \in V \subset U$.

Exercises

1. A path connected space is connected.

2. A connected locally path connected space is also path connected.

3. Every open connected subset of $\mathbf{R}^n$ is path connected.

4. There are examples of path connected subsets of the plane which are not locally path connected.

From now on, assume that $X \subset \mathbf{R}^n$ is path connected and locally path connected.

Let γ_1, γ_2 be two paths in X. Let $I = [0, 1]$. We say that γ_1 and γ_2 are *homotopic* if there is a continuous map $F : I \times I \to X$ such that $F(t, 0) = \gamma_1(t)$ and $F(t, 1) = \gamma_2(t)$ for all $t \in I$. This is written $\gamma_1 \simeq \gamma_2$. One also says that F is a homotopy from γ_1 to γ_2.

Suppose that γ_1 and γ_2 are two paths in X such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$.

We say that γ_1 is homotopic to γ_1 relative to the boundary of I if there is a homotopy $F : I \times I \to X$ such that $F(0, t) = \gamma_1(0) = \gamma_2(0)$ and $F(1, t) = \gamma_1(1) = \gamma_2(1)$ for all $t \in I$. In this case, one writes $\gamma_1 \simeq_\partial \gamma_2$.

A path $\gamma : I \to X$ is called a *loop* if $\gamma(0) = \gamma(1)$.

Let S^1 be the unit circle in the plane (the set $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$. The *standard covering map* from R to S^1 is the map $\pi : \mathbf{R} \to S^1$ defined by

$$\pi(t) = (cos(2\pi t), sin(2\pi t)).$$

In complex notation, using $S^1 = \{z \in \mathbf{C} : \mid z \mid = 1\}$, the map π may be writtne $\pi(t) = exp(2\pi i t)$ with $i = \sqrt{-1}$.

Let X be as above, i.e., X is a connected, locally path connected subspace of some $\mathbf{R}^n$.

Let $f : X \to S^1$ be a continuous map. A *lift* of f to $\mathbf{R}$ is a continuous map $F : X \to \mathbf{R}$ such that $\pi F = f$.

Path Lifting Property of π:

Let $\gamma : I \to S^1$ be a path. Then there is a lift $\tilde{\gamma} : I \to \mathbf{R}$ of γ. Any two such lifts differ by an integer translation. That is, if $F_1 : I \to \mathbf{R}$ and $F_2 : I \to \mathbf{R}$ are paths such that $\pi F_1 = \pi F_2$, then there is an integer n such that $F_1(t) = F_2(t) + n$ for all $t \in I$.

Homotopy Lifting Property of π:

Let γ_1, γ_2 be two paths in S^1 and let $F : I \times I \to S^1$ be a homotopy from γ_1 to γ_2. Then, there is a homotopy $\tilde{F} : I \times I \to \mathbf{R}$ such that $\pi\tilde{F} = F$. Any two such lifts differ by an integer translation.

Thus, the homotopy $\tilde{F}$ is a lift of the homotopy F. If $\tilde{F}_1$ and $\tilde{F}_2$ are two such lifts, then there is an integer n such that $\tilde{F}_1(t, s) = \tilde{F}_2(t, s) + n$ for all $(t, s) \in I \times I$.

There is also a relative version: if $F : \gamma_1 \simeq \gamma_2$ is constant on $\{0\} \times I$ and $\{1\} \times I$, then there is a lift $\tilde{F}$ of F which is also constant on these subsets of $I \times I$.

Definition. A loop is called a *constant* or *constant loop* if its image is a single point. A loop γ in a space X is *homotopic to a constant* or *null-homotopic* if there is a homotopy $F : I \times I$ from γ to a constant loop.

The space X is *simply connected* if it is connected, locally path connected, and every loop γ in X is homotopic to a constant loop.

Examples of simply connected spaces are open and closed balls, the $n-$sphere if $n > 1$, line segments, linear subspaces.

The unit circle S^1 is *not* simply connected.

Proposition 1.1 *Let X be a simply connected space. Then, every continuous map $f : X \to S^1$ has a lift $F : X \to \mathbf{R}$. Any two such lifts differ by the addition of an integer.*

2 Index of a non-vanishing continuous vector field along a continuous curve in the plane

Let Ω be a region in the plane (connected open set). Let $\gamma : I \to \Omega$ be a path in Ω, and let

$$f : Image(\gamma) \to R^2 \setminus \{(0,0)\}$$

be a continuous non-vanishing vector field on the image of γ.

For $z \in Image(\gamma)$, let

$$f_u(z) = \frac{f(z)}{\mid f(z) \mid}$$

be the unit vector determined by $f(z)$.

The map $t \to f_u(\gamma(t))$ is a continuous map from I into S^1. Let $F_u : I \to \mathbf{R}$ be a lift of f_u. Thus, $F_u : I \to \mathbf{R}$ is continuous and $\pi F_u = f_u$.

Definition. The *angular variation of f along γ* is the value $2\pi(F_u(1) - F_u(0))$. We denote this by

$$ang\_var(\gamma, f) \tag{1}$$

Since any two lifts of $t \to f_u(\gamma(t))$ differ by an integer, the quantity $ang\_var(f)$ is independent of the choice of a lift $t \to F_u$.

We have introduced the factor of 2π in the definition of ang_var so that if γ is the positively oriented unit circle and f is its outward normal vector field, then $ang\_var(\gamma, f) = 2\pi$ conforming to standard usage.

Observe that if γ is a loop, then $ang\_var(\gamma, f)$ is $2\pi n$ for some integer n. In this case, we define $Ind(\gamma, f)$ to be this integer. That is, we define

$$Ind(\gamma, f) = \frac{1}{2\pi} ang\_var(\gamma, f)$$

Note that the term *index* is only used for loops. Also, the index removes the factor of 2π so that it becomes an integer in variation of the angle along a loop.

If γ is a C^1 loop and $f(z) = (\eta_1(z), \eta_2(z))$ is a C^2 vector non-vanishing vector field on a neighborhood of the image of γ, then $Ind(\gamma, f)$ can be computed as a line integral:

$$Ind(\gamma, f) = \frac{1}{2\pi} \int_\gamma \frac{\eta_1 d\eta_2 - \eta_2 d\eta_1}{\eta_1^2 + \eta_2^2} \tag{2}$$

Note that, since this is integer valued, numerical integration can be used to calculate $Ind(\gamma, f)$ for quite complicated vector fields f.

3 Homotopy properties of the $Ind(\gamma, f)$

Proposition 3.1 *Let Ω be a region in the plane, and let f be a continuous non-vanishing vector field in Ω. If γ_0 and γ_1 are two loops in Ω which are homotopic through loops, then*

$$Ind(\gamma_0, f) = Ind(\gamma_1, f)$$

Proof. The statement that γ_0 and γ_1 are homotopic through loops means that there is a continuous map

$$F : I \times I \to \Omega$$

such that $F(t, 0) = \gamma_0(t)$ and $F(t, 1) = \gamma_1(t)$ for all $t \in I$, and, $F(0, s) = F(1, s)$ for each $s \in I$.

Thus, if we let $\gamma_s(t)$ be the map defined by $\gamma_s(t) = F(t, s)$, then each map $\gamma_s : I \to \Omega$ is a loop.

Then, the quantity $Ind(\gamma_s, f)$ is defined.

Moreover, the map $s \to Ind(\gamma_s, f)$ is an integer valued continuous map, so it must be constant. QED.

Proposition 3.2 *Let Ω be a region in the plane, and let γ be a loop in Ω. Let f and g be non-vanishing continuous vector fields in Ω so that f and g are homotopic through non-vanishing vector fields along γ. That is, there is a continuous map $F : Image(\gamma) \times I \to \mathbf{R}^2 \setminus \{(0,0\}$ so that*

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x) \quad \forall x \in Image(\gamma).$$

Then,

$$Ind(\gamma, f) = Ind(\gamma, g)$$

Proof. Let

$$F_t : Image(\gamma) \to \mathbf{R}^2 \setminus \{(0,0\}$$

be the map defined by $F_t(x) = F(x,t)$. Thus, each F_t is a non-vanishing vector field along γ.

The map $t \to Ind(\gamma, F_t)$ is continuous and integer valued, so it must be constant. QED.

4 Index of a vector field at an isolated critical point

Let Ω be a region in $\mathbf{R}^2$ and let f be a continuous vector field in Ω. A critical point p of f is a point such that $f(p) = \mathbf{0}$. The critical point p is *isolated* if there is an

$\epsilon > 0$ such that if $y \in \Omega$ is such that $0 < |\, y - p\,| < \epsilon$, then $f(y) \neq 0$. Thus, p is the only critical point of f in the open ball $B_\epsilon(p)$ of radius ϵ centered at p.

A (continuous) Jordan curve C *around* p is a Jordan curve such that p is in the the bounded region of the complement of C.

Let C be a positively oriented continuous Jordan curve around p.

Define the index of the pair (p, f) by

$$Ind(p, f) = Ind(C, f)$$

Observe that, if $-C$ is the negative path of C, then

$$Ind(-C, f) = -Ind(C, f)$$

Remark. Proposition 3.1 yields that if C_1 and C_2 are positively oriented Jordan curves in Ω around p, and C_1 is homotopic to C_2 through loops around p in such a way that f has no critical points on the image of the homotopy, then $Ind(C_1, f) = Ind(C_2, f)$.

5 Formulas for the index

5.1 The Umlaufsatz

Theorem 5.1 *(Umlaufsatz) Let γ be a positively oriented C^1 Jordan curve in the plane parametrized by*

$\gamma : I \rightarrow \mathbf{R}^2$ and assume that $\gamma'(t) \neq \mathbf{0}$ for all $t \in I$ where $\gamma'(t)$ is the tangent vector to γ at $\gamma(t)$.

Then,

$$Ind(\gamma, \gamma') = 1 \tag{3}$$

Proof.

The index does not change if we rotate or translate γ so we may assume that we have chosen the parametrization $\gamma : I \rightarrow R^2$ so that $\gamma(0) = (0,0) = \gamma(1)$, $\gamma(t)$ lies in the upper half plane $\{(x, y) \in \mathbf{R}^2 : y \geq 0\}$ for all $t \in I$, and the unit tangent vector $\frac{\gamma'(0)}{\mid \gamma'(0) \mid}$ is the standard basis vector $(1, 0)$.

Consider the triangle

$$\Delta = \{(s, t) \in \mathbf{R}^2 : 0 \leq s < t \leq 1\}.$$

Define the vector field $\eta(s, t)$ in Δ as follows:

$$\eta(s,t) = \begin{cases} \frac{\gamma(t)-\gamma(s)}{t-s} & \text{if } 0 \leq s < t \leq 1 \\ \gamma'(t) & \text{if } s = t \end{cases}$$

The vector field η is continuous in Δ and vanishes only at $(0, 1)$. Notice that,

$$\lim_{(s,t)\rightarrow(0,1),(s,t)\neq(0,1)} \frac{\eta(s,t)}{\mid \eta(s,t) \mid} = -\frac{\gamma'(0)}{\mid \gamma'(0) \mid}. \tag{4}$$

Thus, the unit vector field η_1 defined by

$$\eta_1(s,t) = \begin{cases} \frac{\eta(s,t)}{\mid \eta(s,t) \mid} & \text{if } (s,t) \neq (0,1) \\ -\frac{\gamma'(0)}{\mid \gamma'(0) \mid} \end{cases}$$

is continuous on Δ.

Consider the four maps $\gamma_0(t) = (t,t)$, $\gamma_1(t) = (0,t)$, $\gamma_2(t,1)$, $\gamma_1\sharp\gamma_2$ from I into Δ.

We have

1. $\gamma_1(1) = \gamma_2(0)$, so $\gamma_1\sharp\gamma_2$ is defined,
2. $\gamma_0(0) = \gamma_1\sharp\gamma_2(0) = (0,0)$, and $\gamma_0(1) = \gamma_1\sharp\gamma_2(1) = (1,1)$, and
3. $Ind(\gamma,\gamma') = \frac{1}{2\pi} ang\_var(\gamma_0, \eta_1)$

Hence, it suffices to prove that

$$ang\_var(\gamma_0, \eta_1) = 2\pi. \tag{5}$$

Proceeding toward this, note that the paths $\gamma_1\sharp\gamma_2$ and γ_0 are homotopic in Δ relative to $\{0,1\}$. Indeed, since Δ is convex, we may take the map $F(t,s) = (1-s)\gamma_1\sharp\gamma_2(t) + s\gamma_0(t)$ as a homotopy from $\gamma_1\sharp\gamma_2$ to γ_0 relative to $\{0,1\}$.

Thus,

$$ang\_var(\gamma_0, \eta_1) = ang\_var(\gamma_1\sharp\gamma_2, \eta_1). \tag{6}$$

But,

$$ang\_var(\gamma_1 \sharp \gamma_2, \eta_1) = ang\_var(\gamma_1, \eta_1) + ang\_var(\gamma_2, \eta_1)$$

Let us consider the vector field η_1 along the two paths γ_1 and γ_2 separately.

The vector field $\eta_1(\gamma_1(t))$ along γ_1 has the following properties.

1. It points into the upper half plane for $0 < t < 1$.
2. $\eta_1(\gamma_1(0))$ is the unit tangent vector $\frac{\gamma'(0)}{\mid \gamma'(0) \mid}$, and
3. $\eta_1(\gamma_1(1))$ is the negative unit tangent vector $-\frac{\gamma'(0)}{\mid \gamma'(0) \mid}$,
4. the map $t \to \eta_1(\gamma_1(1))$ is continuous.

It follows that

$$ang\_var(\gamma_1, \eta_1) = \pi.$$

The vector field $\eta_1(\gamma_2(t))$ along γ_2 has the following properties.

1. It points into the lower half plane for $0 < t < 1$.
2. $\eta_1(\gamma_2(0))$ is the negative unit tangent vector $-\frac{\gamma'(0)}{\mid \gamma'(0) \mid}$, and
3. $\eta_1(\gamma_2(1))$ is unit tangent vector $\frac{\gamma'(0)}{\mid \gamma'(0) \mid}$,

4. the map $t \to \eta_1(\gamma_2(1))$ is continuous.

This implies that $ang\_var(\gamma_2, \eta_1)$ is also π, so formula (5) is satisfied. QED.

6 The index in regions with finitely many critical points

Proposition 6.1 *Let γ be a non-trivial periodic orbit of a C^1 planar vector field. Then, γ is a Jordan curve. Let A be its interior. Then, f has a critical point in A.*

Proof. By the Umlaufsatz, if γ is parametrized so that it is a positively oriented Jordan curve, then $j_f(\gamma) = 1$. Depending on the direction of f with respect to this parametrization, we have $Ind(\gamma, f) = \pm 1$. Let $\bar{A} = A \cup Image(\gamma)$. We use the fact from Topology that $\bar{A}$ is simply connected. Since γ is a periodic orbit for f, it is clear that any critical points of f in $\bar{A}$ must lie in A. Suppose that f has no critical points in A, and let $p \in A$. From simple connectivity, it follows that any two loops in A are homotopic (they are both homotopic to a constant, and homotopy is an equivalence relation). Thus, the curve γ is homotopic to a small positively oriented circle C around p in A. If the circle C is small enough, we have that $Ind(C, f) = 0$ since f is nearly constant on C. But, from Proposition 3.1, Ind(γ,f) = Ind(C,f). This is a contradiction. QED.

Corollary 6.2 *Let f be a C^1 vector field defined on the whole plane $\mathbf{R}^2$, and let $p \in \Omega$ be a point whose forward orbit has compact closure. Then, f has at least one critical point*

Proof. Consider the $\omega-$limit set $\omega(p)$. Since the positive orbit has compact closure, this set is non-empty. If $\omega(p)$ contains a critical point, then we are done. If not, then the Poincaré-Bendixson Theorem gives that $\omega(p)$ is a periodic orbit. Proposition (6.1) gives the existence of a critical point in the interior of $\omega(p)$. QED.

Proposition 6.3 *Let f be a C^1 vector field with only finitely many critical points $x_1, x_2, \ldots, x_n$ in the interior of a positively oriented Jordan curve γ. Then,*

$$j_f(\gamma) = Ind(x_1, f) + \ldots + Ind(x_n, f)$$

Proof. Consider small positively oriented Jordan curves γ_i about x_i in the interior of γ. Join γ to each γ_i by an arc η_i so that the $\eta_i' s$ are disjoint. We may split the curves η_i into small arcs going in opposite directions η_{i1}, η_{i2} and use pieces of γ, γ_i with these new curves to get a simple closed positively oriented curve $\tilde{\gamma}$ whose interior contains no critical points. Thus, $j_f(\tilde{\gamma}) = 0$.

But $j_f(\tilde{\gamma})$ is approximately

$$j_f(\gamma) - \sum Ind(x_i, f).$$

Passing to the limit as the curves η_{ij} approach $\pm\eta_i$, proves the result. QED.

6.1 A geometric formula

Before presenting the geometric formula, we recall that the formula (2) can often be used to calculate $Ind(p, f)$

Now, let us proceed to the geometric formula.

Theorem 6.4 *Let p be an isolated critical point of a continuous vector field f in the region Ω in the plane. Let γ be a positively oriented Jordan curve around p whose interior region is contained in Ω and assume that f has at most finitely many tangencies with γ and they are only interior or exterior tangencies. Let N_i be the number of interior tangencies and N_e be the number of exterior tangencies.*

Then,

$$Ind(p, f) = 1 + \frac{1}{2}(N_i - N_e) \tag{7}$$

This formula is proved in Section 11b of the Lecture Notes.