1. Let
$$\eta(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$
Prove that $\eta(x)$ is a C^∞ function from $\mathbf{R}$ to $\mathbf{R}$.

2. Let $\eta_1(x) = \eta(x)\eta(1-x)$, where η is as in the previous exercise, and let
$$\xi(x) = \frac{\int_0^x \eta_1(t)dt}{\int_0^1 \eta_1(t)dt}$$
The function $\xi(x)$ is a C^∞ function from $\mathbf{R}$ to $\mathbf{R}$ such that $\xi(x) \geq 0\ \forall x, \xi(x) = 0$ for $x \leq 0$, $\xi(x) = 1$ for $x \geq 1$. and $\int_{-\infty}^{\infty} \xi(t)dt = 1$. Using a modification of ξ prove that, for any $a < b$, there is a C^∞ function $\rho : \mathbf{R} \to \mathbf{R}$ such that $\rho(x) \geq 0\ \forall x, \rho(x) = 0$ for $x \leq a$, and $\rho(x) = 1$ for $x \geq b$.

3. (a) Let $x \neq y$ be distinct points in $\mathbf{R}^n$. Prove that there is a C^∞ function ρ from $\mathbf{R}^n$ to $\mathbf{R}$ such that $\rho(u) \in [0, 1]\ \forall u$, $\rho(x) = 0$ and $\rho(y) = 1$.

   (b) Let γ_1 and γ_2 be distinct circles in the plane $\mathbf{R}^2$. Prove that there is a C^∞ function ρ from $\mathbf{R}^2$ to $\mathbf{R}$ such that $\rho(x) \geq 0$ for all x, $\rho(x) = 0$ for $x \in \gamma_1$, and $\rho(x) = 1$ for $x \in \gamma_2$.

4. Show that according as $ad - bc > 0$ or $ad - bc < 0$, the index of the origin with respect to the linear vector field $f_0(x, y) = (ax+by, cx+dy)$ is ± 1.

5. Suppose that $f(x, y) = (f_1(x, y), f_2(x, y))$ is a C^1 vector field with an isolated critical point at $0 \in \mathbf{R}^2$ and the derivative of f at 0 is the linear map f_0 in exercise 4. Show that if $ad - bc > 0$, then the index of f at 0 is +1 while if $ad - bc < 0$, then the index at 0 of f is -1.

6. Let $f(z) = z^k$ where $z = x + iy$ and z^k means the complex number z is multiplied by itself k-times. Consider f as a vector field in $\mathbf{R}^2$. Show that the index of f at 0 is k.

7. Let $f(z) = \bar{z}^k$ where $z = x + iy$ and $\bar{z}^k$ means the complex conjugate of z multiplied by itself k times. Consider f as a vector field in $\mathbf{R}^2$. Show that the index of f at 0 is $-k$. Recall that if $z = x + iy$, then $\bar{z} = x - iy$ where $i = \sqrt{-1}$.

8. Let (X, d) be a compact metric space. A map $T : X \to X$ is an *isometry* if $d(Tx, Ty) = d(x, y)$ for all $x, y \in X$. Suppose that $T : X \to X$ is an isometry such that there is some $x_0 \in X$ whose orbit is dense in X. Prove that for any $y \in X$, both the forward and backward orbits of y are dense in X.

9. A function $\phi : \mathbf{R} \to \mathbf{R}$ is *periodic* if there is a positive number $\tau > 0$ such that $\phi(x + \tau) = \phi(x)$ for all $x \in \mathbf{R}$. The number τ is called a *period* of ϕ. Let $S^1 = \{z \in \mathbf{C} : | z | = 1\}$ be the unit circle in the complex plane, and let $\rho : \mathbf{R} \to S^1$ be the standard covering projection. Let $C(S^1, S^1)$ be the collection of continuous self-maps of S^1, and let $C(\mathbf{R}, \mathbf{R})$ be the set of continuous self-maps of $\mathbf{R}$. For an element $f \in C(S^1, S^1)$, a *lift* of f to $C(\mathbf{R}, \mathbf{R})$ is an element $F \in C(\mathbf{R}, \mathbf{R})$ such that $\rho F = f\rho$.

   Let d be an integer. Prove that F in $C(\mathbf{R}, \mathbf{R})$ is a lift of a map $f \in C(S^1, S^1)$ of degree d if and only if there is a periodic function $\phi : \mathbf{R} \to \mathbf{R}$ of period 1 such that $F(x) = d\, x + \phi(x)$ for all $x \in \mathbf{R}$.