1. Let 
\[ \eta(x) = \begin{cases} 
    e^{-\frac{1}{x^2}} & \text{if } x > 0 \\
    0 & \text{if } x \leq 0 
\end{cases} \]
Prove that \( \eta(x) \) is a \( C^\infty \) function from \( \mathbb{R} \) to \( \mathbb{R} \).

2. Let \( \eta_1(x) = \eta(x)(1 - x) \), where \( \eta \) is as in the previous exercise, and let
\[ \xi(x) = \frac{\int_x^1 \eta_1(t)dt}{\int_0^1 \eta_1(t)dt} \]
The function \( \xi(x) \) is a \( C^\infty \) function from \( \mathbb{R} \) to \( \mathbb{R} \) such that \( \xi(x) \geq 0 \forall x, \xi(x) = 0 \) for \( x \leq 0 \), \( \xi(x) = 1 \) for \( x \geq 1 \). Using a modification of \( \xi \) prove that, for any \( a < b \), there is a \( C^\infty \) function \( \rho \) from \( \mathbb{R} \) to \( \mathbb{R} \) such that \( \rho(x) \geq 0 \forall x, \rho(x) = 0 \) for \( x \leq a \), and \( \rho(x) = 1 \) for \( x \geq b \).

3. (a) Let \( x \neq y \) be distinct points in \( \mathbb{R}^n \). Prove that there is a \( C^\infty \) function \( \rho \) from \( \mathbb{R}^n \) to \( \mathbb{R} \) such that \( \rho(u) \in [0,1] \forall u, \rho(x) = 0 \) and \( \rho(y) = 1 \).

(b) Let \( \gamma_1 \) and \( \gamma_2 \) be distinct circles in the plane \( \mathbb{R}^2 \). Prove that there is a \( C^\infty \) function \( \rho \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \) such that \( \rho(x) \geq 0 \forall x, \rho(x) = 0 \) for \( x \in \gamma_1 \), and \( \rho(x) = 1 \) for \( x \in \gamma_2 \).

4. Show that according as \( ad - bc > 0 \) or \( ad - bc < 0 \), the index of the origin with respect to the linear vector field \( f_0(x,y) = (ax + by, cx + dy) \) is \( \pm 1 \).

5. Suppose that \( f(x,y) = (f_1(x,y), f_2(x,y)) \) is a \( C^1 \) vector field with an isolated critical point at \( 0 \in \mathbb{R}^2 \) and the derivative of \( f \) at 0 is the linear map \( f_0 \) in exercise 4. Show that if \( ad - bc > 0 \), then the index of \( f \) at 0 is +1 while if \( ad - bc < 0 \), then the index at 0 of \( f \) is -1.

6. Let \( f(z) = z^k \) where \( z = x + iy \) and \( z^k \) means the complex number \( z \) is multiplied by itself \( k \)-times. Consider \( f \) as a vector field in \( \mathbb{R}^2 \). Show that the index of \( f \) at 0 is \( k \).

7. Let \( f(z) = \bar{z}^k \) where \( z = x + iy \) and \( \bar{z}^k \) means the complex conjugate of \( z \) multiplied by itself \( k \) times. Consider \( f \) as a vector field in \( \mathbb{R}^2 \). Show that the index of \( f \) at 0 is \( -k \). Recall that if \( z = x + iy \), then \( \bar{z} = x - iy \) where \( i = \sqrt{-1} \).
8. Let $(X,d)$ be a compact metric space. A map $T : X \to X$ is an isometry if $d(Tx,Ty) = d(x,y)$ for all $x, y \in X$. Suppose that $T : X \to X$ is an isometry such that there is some $x_0 \in X$ whose orbit is dense in $X$. Prove that for any $y \in X$, both the forward and backward orbits of $y$ are dense in $X$.

9. A function $\phi : \mathbb{R} \to \mathbb{R}$ is periodic if there is a positive number $\tau > 0$ such that $\phi(x + \tau) = \phi(x)$ for all $x \in \mathbb{R}$. The number $\tau$ is called a period of $\phi$. Let $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ be the unit circle in the complex plane, and let $\rho : \mathbb{R} \to S^1$ be the standard covering projection. Let $C(S^1, S^1)$ be the collection of continuous self-maps of $S^1$, and let $C(\mathbb{R}, \mathbb{R})$ be the set of continuous self-maps of $\mathbb{R}$. For an element $f \in C(S^1, S^1)$, a lift of $f$ to $C(\mathbb{R}, \mathbb{R})$ is an element $F \in C(\mathbb{R}, \mathbb{R})$ such that $\rho F = f \rho$.

Let $d$ be an integer. Prove that $F$ in $C(\mathbb{R}, \mathbb{R})$ is a lift of a map $f \in C(S^1, S^1)$ of degree $d$ if and only if there is a periodic function $\phi : \mathbb{R} \to \mathbb{R}$ of period 1 such that $F(x) = d \cdot x + \phi(x)$ for all $x \in \mathbb{R}$.