1. Find the general real solution to the differential equation $\dot{x} = Ax$ for each of the following matrices $A$. Also, draw a rough sketch of the orbits in each case.

(a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Let

$$\eta(x) = \begin{cases} e^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Prove that $\eta(x)$ is a $C^\infty$ function from $\mathbb{R}$ to $\mathbb{R}$.

3. Let $\eta_1(x) = \eta(x)\eta(1-x)$, where $\eta$ is as in the previous exercise, and let

$$\xi(x) = \frac{\int_x^1 \eta_1(t)dt}{\int_0^1 \eta_1(t)dt}$$

The function $\xi(x)$ is a $C^\infty$ function from $\mathbb{R}$ to $\mathbb{R}$ such that $\xi(x) \geq 0 \forall x, \xi(x) = 0$ for $x \leq 0$, and $\xi(x) = 1$ for $x \geq 1$. Using a modification of $\xi$ prove that, for any $a < b$, there is a $C^\infty$ function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(x) \geq 0 \forall x, \rho(x) = 0$ for $x \leq a$, and $\rho(x) = 1$ for $x \geq b$. 
4. (a) Let $x \neq y$ be distinct points in $\mathbb{R}^n$. Prove that there is a $C^\infty$ function $\rho$ from $\mathbb{R}^n$ to $\mathbb{R}$ such that $\rho(u) \in [0, 1]$ $\forall u$, $\rho(x) = 0$ and $\rho(y) = 1$.

(b) Let $\gamma_1$ and $\gamma_2$ be distinct circles in the plane $\mathbb{R}^2$, both of which centered at the origin. Prove that there is a $C^\infty$ function $\rho$ from $\mathbb{R}^2$ to $\mathbb{R}$ such that $\rho(x) \geq 0$ for all $x$, $\rho(x) = 0$ for $x \in \gamma_1$, and $\rho(x) = 1$ for $x \in \gamma_2$.

5. Show that according as $ad - bc > 0$ or $ad - bc < 0$, the index of the origin with respect to the linear vector field $f_0(x, y) = (ax+by, cx+dy)$ is $\pm 1$.

6. Suppose that $f(x, y) = (f_1(x, y), f_2(x, y))$ is a $C^1$ vector field with an isolated critical point at $0 \in \mathbb{R}^2$ and the derivative of $f$ at $0$ is the linear map $f_0$ in exercise 5. Show that if $ad - bc > 0$, then the index of $f$ at $0$ is $+1$ while if $ad - bc < 0$, then the index at $0$ of $f$ is $-1$.

7. Let $f(z) = z^k$ where $z = x + iy$ and $z^k$ means the complex number $z$ is multiplied by itself $k$-times. Consider $f$ as a vector field in $\mathbb{R}^2$. Show that the index of $f$ at $0$ is $k$.

8. Let $f(z) = \bar{z}^k$ where $z = x + iy$ and $\bar{z}^k$ means the complex conjugate of $z$ multiplied by itself $k$ times. Consider $f$ as a vector field in $\mathbb{R}^2$. Show that the index of $f$ at $0$ is $-k$. Recall that if $z = x + iy$, then $\bar{z} = x - iy$ where $i = \sqrt{-1}$.

9. Give an example of a $C^\infty$ vector field $f$ in the plane which has the unit circle as the only non-trivial closed orbit.

Hint: Consider the use of polar coordinates.