1. Suppose $T : X \to Y$ is a linear map of a Banach space X into Banach space Y. Let

$$\begin{aligned} A &= \inf\{k : \mid Tx \mid \leq k \mid x \mid \forall\, x \in X\} \\ B &= \sup_{x \neq 0} \left\{ \frac{\mid Tx \mid}{\mid x \mid} \right\} \\ C &= \sup_{\mid x \mid \leq 1} \{\mid Tx \mid\} \end{aligned}$$

   Show that $A = B = C$.

2. Suppose $|\cdot|_1, |\cdot|_2$ are two norms on $\mathbf{R}^n$. Prove that there are constants $C_1 > 0, C_2 > 0$ such that for every $x \in \mathbf{R}^n$,

$$C_1 \mid x \mid_1 \leq \mid x \mid_2 \leq C_2 \mid x \mid_1$$

3. Suppose $T : X \to Y$ is a one-to-one continuous onto linear map from the Banach space X to the Banach space Y and there is a constant $k > 0$ such that $\mid Tx \mid \geq k$ for all $\mid x \mid = 1$. Prove that there is a unique continuous linear map $S : Y \to X$ such that $S(Tx) = x$ for all x. For those who know some functional analysis: is the same conclusion true for one-to-one continous onto linear maps without the assumption that there is such a k?

4. Prove that every linear map from $\mathbf{R}^n$ to $\mathbf{R}^n$ is uniformly continuous.

5. Let $I = [0, 1]$ be the closed unit interval. Show that the closed unit ball in the Banach space $\mathcal{C}(I, \mathbf{R}^n)$ is not compact.

6. Show that the Schauder Fixed Point Theorem becomes false if either of the compactness or convexity conditions does not hold.

7. A compact topological space X has the *fixed point property* or *fpp* if every continuous self-map of X has a fixed point. Prove that this property is preserved by homeomorphism. That is, if Y is homeomorphic to X and X has the fpp, then Y also has the fpp.

8. Let $f : [0, 1] \to [0, 1]$ be continuous. Show that, given $\epsilon > 0$, there is a continuous $g : [0, 1] \to [0, 1]$ such that g has only finitely many fixed points and $\mid f(x) - g(x) \mid < \epsilon$ for all $x \in [0, 1]$.

9. Let F be an arbitrary closed subset of $I = [0,1]$. Show that there is a strictly increasing continuous function ϕ from I to I such that the set of fixed points of ϕ is precisely F.

10. Consider the norms $|\cdot|_p, |\cdot|_\infty$ on $\mathbf{R}^n$ defined by

$$| x |_p = \left(\sum_{1 \le i \le n} | x_i |^p \right)^{\frac{1}{p}},$$

$$| x |_\infty = \sup_{1 \le i \le n} | x_i |$$

where $p > 1$.

Prove that

$$\lim_{p \to \infty} | x |_p = | x |_\infty.$$