1. Suppose $T : X \to Y$ is a linear map of a Banach space $X$ into Banach space $Y$. Let

$$A = \inf \{ k : |Tx| \leq k|x| \forall x \in X \}$$
$$B = \sup_{x \neq 0} \left\{ \frac{|Tx|}{|x|} \right\}$$
$$C = \sup_{|x| \leq 1} \{|Tx|\}$$

Show that $A = B = C$.

2. Suppose $|\cdot|_1, |\cdot|_2$ are two norms on $\mathbb{R}^n$. Prove that there are constants $C_1 > 0, C_2 > 0$ such that for every $x \in \mathbb{R}^n$,

$$C_1 |x|_1 \leq |x|_2 \leq C_2 |x|_1$$

3. Suppose $T : X \to Y$ is a one-to-one continuous onto linear map from the Banach space $X$ to the Banach space $Y$ and there is a constant $k > 0$ such that $|Tx| \geq k$ for all $|x| = 1$. Prove that there is a unique continuous linear map $S : Y \to X$ such that $S(Tx) = x$ for all $x$. For those who know some functional analysis: is the same conclusion true for one-to-one continuous onto linear maps without the assumption that there is such a $k$?

4. Prove that every linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is uniformly continuous.

5. Let $I = [0,1]$ be the closed unit interval. Show that the closed unit ball in the Banach space $\mathcal{C}(I, \mathbb{R}^n)$ is not compact.

6. Show that the Schauder Fixed Point Theorem becomes false if either of the compactness or convexity conditions does not hold.

7. A compact topological space $X$ has the fixed point property or fpp if every continuous self-map of $X$ has a fixed point. Prove that this property is preserved by homeomorphism. That is, if $Y$ is homeomorphic to $X$ and $X$ has the fpp, then $Y$ also has the fpp.
8. Let \( f : [0, 1] \rightarrow [0, 1] \) be continuous. Show that, given \( \epsilon > 0 \), there is a continuous \( g : [0, 1] \rightarrow [0, 1] \) such that \( g \) has only finitely many fixed points and \( |f(x) - g(x)| < \epsilon \) for all \( x \in [0, 1] \).

9. Let \( F \) be an arbitrary closed subset of \( I = [0, 1] \). Show that there is a strictly increasing continuous function \( \phi \) from \( I \) to \( I \) such that the set of fixed points of \( \phi \) is precisely \( F \).

10. Consider the norms \( |\cdot|_p, |\cdot|_\infty \) on \( \mathbb{R}^n \) defined by

\[
| x |_p = \left( \sum_{1 \leq i \leq n} |x_i|^p \right)^{\frac{1}{p}},
\]

\[
| x |_\infty = \sup_{1 \leq i \leq n} |x_i|
\]

where \( p > 1 \).

Prove that

\[
\lim_{p \to \infty} | x |_p = | x |_\infty.
\]