7. Some Special Second Order Equations

There are certain second order differential equations, even non-linear, which reduce to first order equations. We will describe some of these now.

Type 1:

\[ y'' = f(x, y'). \]

Here the variable \( y \) is missing from the right hand side.

We proceed as follows.

Set \( v = y' \). We get

\[ y'' = v' = f(x, v) \]

Thus, we get a first order d.e. for \( v \). If we can use our known methods to solve this, then we get \( y \) by integrating \( v \).

Example 1:

\[ y'' = x(y')^2 \]

Set \( v = y' \). Then,

\[ y'' = v' = xv^2 \]

is a separable d.e. We solve it.

\[ \frac{dv}{v^2} = xdx \]

\[ \frac{-1}{v} = \frac{x^2}{2} + C \]

\[ v = \frac{1}{\frac{x^2}{2} - C} \]

\[ = \frac{1}{C - \frac{x^2}{2}} \quad \text{different C} \]

\[ = \frac{2}{C_1^2 - x^2} \]

\[ = \frac{1}{C_1(C_1 + x)} + \frac{1}{C_1(C_1 - x)} \]
So,

\[ y' = \frac{1}{C_1(C_1 + x)} + \frac{1}{C_1(C_1 - x)} \]

which gives

\[ y = \frac{1}{C_1} \log(C_1 + x) - \frac{1}{C_1} \log(C_1 - x) + C_2 \]

as the general solution.

**Type 2:**

\[ y'' = f(y, y'). \]

Here the independent variable is missing. Again, we set \( v = y' \) and get

\[ v' = f(y, v). \]

We try to treat \( y \) as a new independent variable. Then,

\[ v' = \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = \frac{dv}{dy} v. \]

The equation becomes

\[ y'' = v' = \frac{dv}{dy} v = f(y, v), \]

or

\[ \frac{dv}{dy} = \frac{1}{v} f(y, v). \]

We solve this, and then integrate \( y = v' \) to get \( y \).

**Example:**

\[ y'' = yy' \]

Setting \( y' = v \), we get

\[ y'' = v_y v = yv \]
So,

\[ v_y = y \]

or,

\[ dv = ydy \]

\[ v = \frac{y^2}{2} + C \]

\[ y' = \frac{y^2}{2} + C' \]

\[ \frac{dy}{\frac{y^2}{2} + C'} = dx \]

Then, we solve for \( y(x) \) as before.