5a. Some Additional Examples of Exact Equations and Integrating Factors

Some worked problems on Exact Equations and Integrating Factors

Example 1

Find the general solution $F(x, y) = C$ to the differential equation

$$(12 \cdot x^2 \cdot y^4 - 6 \cdot x^2) \cdot dx + (16 \cdot x^3 \cdot y^3 + 8 \cdot y^3) \cdot dy = 0$$

Solution:

Here,

$M = 12 \cdot x^2 \cdot y^4 - 6 \cdot x^2, \quad N = 16 \cdot x^3 \cdot y^3 + 8 \cdot y^3$

Check for Exactness:

$M_y = 48 \cdot x^2 \cdot y^3, \quad N_x = 48 \cdot x^2 \cdot y^3$

So, it is exact.

Set

$F = \int^x M \, dx + g(y) = 12 \cdot (x^3/3) \cdot y^4 - 6 \cdot (x^3/3) + g(y)$

Take $F_y$ and set it equal to $N$:

$F_y = 12 \cdot (x^3/3) \cdot 4 \cdot y^3 + g'(y) = 16 \cdot x^3 \cdot y^3 + 8 \cdot y^3$

$g'(y) = 8 \cdot y^3$

So,

$F(x, y) = 12 \cdot (x^3/3) \cdot y^4 - 6 \cdot (x^3/3) + g(y)$

$= 12 \cdot (x^3/3) \cdot y^4 - 6 \cdot (x^3/3) + 8 \cdot y^4/4$

$= 4 \cdot x^3 \cdot y^4 - 2 \cdot x^3 + 2 \cdot y^4$
Solution:
\[ F(x, y) = 4 \cdot x^3 \cdot y^4 - 2 \cdot x^3 + 2 \cdot y^4 \]

Example 2
Find the general solution \( F(x, y) = C \) to the differential equation
\[
(8 \cdot x \cdot y^4 + 12 \cdot y^4 - 6) \cdot dx + (8 \cdot x^2 \cdot y^3 + 16 \cdot x \cdot y^3) \cdot dy = 0
\]

Solution:
Here,
\[
M = (8 \cdot x \cdot y^4 + 12 \cdot y^4 - 6), \quad N = 8 \cdot x^2 \cdot y^3 + 16 \cdot x \cdot y^3
\]
Check for Exactness:
\[
M_y = 32 \cdot x \cdot y^3 + 48 \cdot y^3, \quad N_x = 16 \cdot x \cdot y^3 + 16 \cdot y^3
\]
\[
M_y - N_x = 16 \cdot x \cdot y^3 + 32 \cdot y^3
\]
Not exact.
Check for integrating factor \( \mu(x) \):
\[
\frac{M_y - N_x}{N} = \frac{16y^3(x + 2)}{8xy^3(x + 2)} = \frac{2}{x}
\]
So, there is an integrating factor of the form
\[
exp\left(\int \frac{2}{x} \right) = x^2
\]
So,
\[
x^2 \cdot (Mdx + Ndy) = 0 \text{ is exact}
\]
So,
\[
(8x^3y^4 + 12x^2y^4 - 6x^2)dx + (8x^4y^3 + 16x^3y^3)dy = 0
\]
is exact.
Solve as in Example 1:

\[ F(x, y) = \int x (8x^3y^4 + 12x^2y^4 - 6x^2)dx + g(y) \]
\[ = 8(x^4/4)y^4 + 12(x^3/3)y^4 - 6x^3/3 + g(y) \]
\[ = 2x^4y^4 + 4x^3y^4 - 2x^3 + g(y) \]

\[ F_y = 32 \times (x^4/4)y^3 + 48 \times (x^3/3)y^3 + g'(y) \]
\[ = 8x^4y^3 + 16x^3y^3 + g'(y) \]
\[ = N \]
\[ = 8x^4y^3 + 16x^3y^3 \]
\[ \implies g'(y) = 0 \]
\[ \implies g(y) = C_1 \]

Since \( g(y) \) is constant, it can be absorbed into the right side of \( F(x, y) = C \).
Solution:

\[ F(x, y) = 2x^4y^4 + 4x^3y^4 - 2x^3 \]

**Example 3**
Find the general solution \( F(x, y) = C \) to the differential equation

\[ (8x^3y^2 + 12x^2y^2)dx + (8x^4y + 16x^3y + 9)dy = 0 \]

\[ F(x, y) = \]

Solution:
Here,

\[ M = 8x^3y^2 + 12x^2y^2, \quad N = 8x^4y + 16x^3y + 9 \]
Try $M_y - N_x$:

$$M_y - N_x = -16x^3y - 24x^2y \neq 0$$

$$\frac{M_y - N_x}{N} \text{ not a function of } x$$

Try

$$\frac{M_y - N_x}{M} = \frac{-16x^3y - 24x^2y}{8x^3y^2 + 12x^2y^2}$$

$$= \frac{-2(8x^3y + 12x^2y)}{y(8x^3y + 12x^2y)}$$

$$= \frac{-2}{y}$$

So, can find an integrating factor of the form

$$\mu(y) = exp\left(\int -\frac{M_y - N_x}{M} \, dy\right) = exp\left(\int -\left(-\frac{2}{y}\right)\right) = exp\left(\int \frac{2}{y}\right) = y^2$$

Now, we proceed as before obtaining the exact equation

$$(8x^3y^4 + 12x^2y^4)dx + (8x^4y^3 + 16x^3y^3 + 9y^2)dy = 0$$

and we can solve this as before.

$$M = 8x^3y^4 + 12x^2y^4, N = 8x^4y^3 + 16x^3y^3 + 9y^2$$

$$F(x, y) = \int M + g(y)$$

$$= 8(x^4/4) * y^4 + 12(x^3/3) * y^4 + g(y)$$

$$F_y = 8x^4y^3 + 16x^3y^3 + g'(y)$$

$$= N$$

$$= 8x^4y^3 + 16x^3y^3 + 9y^2$$

$$\Rightarrow g'(y) = 9y^2$$

$$\Rightarrow g(y) = 3y^3$$