3. Separable differential Equations

A differential equation of the form
\[
\frac{dy}{dx} = f(x, y)
\]
is called \textit{separable} if the function \( f(x, y) \) decomposes as a product \( f(x, y) = \phi_1(x)\phi_2(y) \) of two functions \( \phi_1 \) and \( \phi_2 \).

Proceeding formally we can rewrite this as
\[
\frac{dy}{\phi_2(y)} = \phi_1(x)dx.
\]

Using the second formula, we can integrate both sides to get
\[
\int y \frac{dy}{\phi_2(y)} = \int x \phi_1(x)dx + C.
\]
as the general solution.

Note that this is an implicit relation between $y(x)$ and $x$.

Indeed, the last integral formula has the form

$$F(y(x)) = G(x) + C$$

for some functions $F$ and $G$. To find $y(x)$ as a function of $x$ we would have to solve this implicit relationship.

This is frequently hard to do, so we will leave the solution in implicit form.

A more general version of this is the d.e.

$$M(x)dx + N(y)dy = 0 \quad (1)$$

We say that the general solution to this d.e. is an expression
\[ f(x, y) = C \]

where \( f_x = M(x) \), \( g_y = M(y) \).

Since the family of the preceding equation as \( C \) varies is a family of curves, one sometimes says that this is the family of integral curves for the d.e. (1).

Also, the initial value problem

\[ \frac{dy}{dx} = \phi_1(x)\phi_2(y), \ y(x_0) = y_0 \]

can be solved as

\[ \int_{y_0}^{y} \frac{dy}{\phi_2(y)} = \int_{x_0}^{x} \phi_1(x)dx \]

This picks out a specific curve in the family of integral curves.

Examples:

1. Find the general solution of the d.e.
\[ \frac{dy}{dx} = \frac{x^2}{1 - y^2}. \]

Write this as
\[ -x^2 \, dx + (1 - y^2) \, dy = 0 \]

The general solution has the form
\[ f(x, y) = C \]

where \( f_x = -x^2 \) and \( f_y = 1 - y^2 \).

Hence, we can take

\[ f = \int x \, -x^2 \, dx + \int y \, (1 - y^2) \, dy \]
\[ = \frac{x^3}{3} + y - \frac{y^3}{3} \]

and the general solution is
\[ -\frac{x^3}{3} + y - \frac{y^3}{3} = C. \]
2. For the preceding d.e. find the integral curve passing through (1, 3).

We need to substitute \( x = 1, \ y = 3 \) in the above formula.

We get

\[
\frac{1}{3} + 3 - \frac{81}{3} = C,
\]

so the desired curve is

\[
-x^2/3 + y - y^3/3 = \frac{10}{3} - \frac{81}{3}.
\]

3. Solve the IVP

\[
\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad y(0) = -1.
\]

Write this as

\[-(3x^2 + 4x + 2)\,dx + 2(y - 1)\,dy = 0.\]

Integrate to
\[-x^3 - 2x^2 - 2x + y^2 - 2y = C,\]

and plug in \(x = 0, y = -1\) to get \(C = 3\).

So,

\[
\text{ANS: } -x^3 - 2x^2 - 2x + y^2 - 2y = 3.
\]

A difference between linear and non-linear first order scalar equations.

Given a first order linear d.e. of the form

\[
y' + p(t)y = g(t)
\]

with \(p(t), g(t)\) continuous on an interval \(I\), and a point \(t_0\) in \(I\), the solution to the IVP

\[
y' + p(t)y = g(t), \ y(t_0) = y_0
\]
exists on the whole interval $I$. This fails for non-linear equations. As an example, take

$$y' = y^2, \ y(0) = y_0$$

We solve this equation as

$$\frac{dy}{y^2} = dt$$

$$\int y \frac{dy}{y^2} = \int_0^t dt$$

$$-\frac{1}{y} = t + C$$

$$y = -\frac{1}{t + C}, = y_0 = -\frac{1}{C}$$

This solution blows up at the point $t = -C$. The graphs of solutions look like those in the following figure.
In[8]:= C = 1; Plot[-1/(t + 1), {t, -2, 2}]

Out[8]= -Graphics-

In[9]:= C = -1; Plot[-1/(t - 1), {t, -2, 2}]

Out[9]= -Graphics-