2a. Bernoulli’s Differential Equation

A differential equation of the form

\[ y' + p(t)y = g(t)y^n \]  \hspace{1cm} (1)

is called Bernoulli’s differential equation.

If \( n = 0 \) or \( n = 1 \), this is linear. If \( n \neq 0, 1 \), we make the change of variables \( v = y^{1-n} \). This transforms (1) into a linear equation.

Let us see this.

We have

\[ v = y^{1-n} \]

\[ v' = (1 - n)y^{-n}y' \]

\[ y' = \frac{1}{1 - n}y^n v' \]

and

\[ y = y^n v \]

Hence,

\[ y' + py = gy^n \]

becomes

\[ \frac{1}{1 - n}y^n v' + py^n v = gy^n \]
Dividing \(y^n\) through and multiplying by \(1 - n\) gives

\[v' + (1 - n)pv = (1 - n)g.\]

We can then find \(v\) and, hence, \(y = v^{\frac{1}{1-n}}\).

Example.

Find the general solution to

\[y' + ty = ty^3.\]

We put \(v = y^{-2}\)

We get

\[v' = (-2)y^{-3}y',\; y = y^3v\]

So,

\[y' + ty = ty^3\]
\[(-1/2)y^3v' + ty^3v = ty^3\]
\[v' - 2tv = -2t\]
\[\mu = e^{-t^2}\]

\[v = e^{t^2} \left( \int e^{-t^2} (-2t) dt + c \right)\]
\[= e^{t^2} \left( e^{-t^2} + c \right)\]
\[= 1 + ce^{t^2},\]
and,

\[ y = v^{-\frac{1}{2}} = \left[ 1 + ce^{t^2} \right]^{-\frac{1}{2}}. \]
A Bernoulli IVP

Solve \( \frac{dy}{dx} + 5y = 2x^2y^4 \), \( y(1) = 3 \)

or \( y' + \frac{5}{x} y = 2x y^4 \)

Bernoulli with \( n = 4 \)

Let \( u = y^{1-n} = y^{-3} \)

Get \( u' - \frac{15}{x} u = -6x \) linear

\( p = -\frac{15}{x} \)

\( m = e^{\int p \, dx} = e^{-15 \log(x)} = x^{-15} \)

\( u = x^{15} \left[ \int -6x^{-14} \, dx + C \right] \)

\( = x^{15} \left( -6 \cdot \frac{x^{-13}}{-13} + C \right) = \frac{6}{13} x^2 + (C x^{15} - \frac{6}{13}) \)

\( u(1) = 3 = \frac{1}{27} \)

Use \( u(1) = \frac{1}{27} \): \( \frac{6}{13} + C = \frac{1}{27} \), \( C = \frac{1}{27} - \frac{6}{13} \)

\( y = \left[ \frac{6}{13} x^2 + \left( \frac{1}{27} - \frac{6}{13} \right) x^{15} \right]^{-\frac{1}{3}} \)