Partial Differential Equations – Separation of Variables

1 Partial Differential Equations and Operators

Let $\mathcal{C} = \mathcal{C}(R^2)$ be the collection of infinitely differentiable functions from the plane to the real numbers $\mathbf{R}$, and let r be a positive integer.

Consider the three operators from $\mathcal{C}$ to $\mathcal{C}$ defined by

$$u \to \frac{\partial^r u}{\partial t^r}, u \to \frac{\partial^s u}{\partial x^s}, u \to \frac{\partial^{r+s} u}{\partial t^r \partial x^s}$$

We call these *simple partial differential operators* on $\mathcal{C}$.

We also denote them by $\partial_t^r u, \partial_x^s u, \partial_{tx}^{r,s} u$, respectively.

To save writing, we use the notation PD0 to denote the words *partial differential operator*.

The *order* of the simple PDO is the sum of its exponents.

When the order is small, we use the standard notation $u_{ttt} = \partial_t^3, u_{txx} = \partial_{tx}^{1,2}$, etc.

Note that simple PDO's are *linear operators* on $\mathcal{C}$.

Let n be a positive integer. An $n-$th order partial differential operator on $\mathcal{C}$ is a map $F : \mathcal{C} \to \mathcal{C}$ which can be written as a finite sum

$$F(u) = \sum_{i=1}^{n} \alpha_i(x,t) F_i(u)$$

where

1. each $\alpha_i(x,t)$ is in $\mathcal{C}$, (these are called the *coefficients* of F),
2. each $F_i(u)$ is a simple PDO of order less than or equal to n, and
3. at least one of the F_i has order n.

Let us give some examples.

$u_{ttt} + 2u_{tx} - u_{xxxx}$	order 4
$u_{tt} + 3xt^{10}\ u_{xxx}$	order 3
$x^2t^2\ u_x + u_t$	order 1

If the $\alpha_i(t,x)$ are constants, we say the operator F has *constant coefficients.*

We use the term *partial differential operator* or *PDO* for one of any order greater than 0.

Note that, sometimes, we use the variable x, y on the plane instead of t, x, as in $u_{xx} + u_{yy}$.

Given a PDO, F we get an associated partial differential equation, denoted PDE, by

$$F(u) = h(x,t)$$

where $h(x,t)$ is a function of x and t.

For the remainder of this section we assume that $h(x,t) = 0$; i.e, that the PDE is *homogeneous*

The PDO is called *linear* if it is linear as a map from $\mathcal{C}$ to itself.

The set of solutions (if they exist) of a linear homogeneous PDE forms a linear subset of $\mathcal{C}$. That is, the superposition principle holds: if u, v are solutions, and α, β are scalars, then $\alpha u + \beta v$ is also a solution.

A PDE is called *separable* if it can be written as

$$\psi(t)\partial_t^r u + \phi(x)\partial_x^s u = 0 \tag{1}$$

for some positive integers r, s.

where ∂_t^r involves only partial derivatives of u with respect to t and $\partial_x^s u$ involves only partial derivatives with respect to x.

Examples

- $(t^2+1)u_{ttt} - 25exp(x)u_{xx} = 0$ and $sin(3x)u_{ttt} - 25(t^3+3t)u_{xx} = 0$ are separable, but
- $(t+x)^3u_t - (exp(2x+t)u_x = 0$ is *not* separable

2 Separable PDE's

Recall that at the beginning of the course, we studied separable first order ordinary differential equations.

These had the form

$$\frac{dy}{dx} = \phi(x)\psi(y)$$

or, in differential form

$$dy = \phi(x)\psi(y)dx, \ dy - \phi(x)\psi(y)dx = 0$$

Let us say that a PDE is *separable* if it can be written as

$$\partial_t u = \psi(t)\phi(x)\partial_x u \tag{2}$$

for some functions $\psi(t)$ and $\phi(x)$.

Separable PDE's can be reduced to two ODE's by what is called the **Method of Separation of Variables**.

The method is very simple.

We try to find a solution of the form $u(x,t) = T(t)X(x)$ which is a product of a function of t and a function of x.

Letting DT denote derivatives of T with respect to t and DX denote derivatives of X with respect to x, we have

$$\begin{aligned} \partial_t u &= DT \cdot X \\ \partial_x u &= T \cdot DX, \end{aligned}$$

so,

$$\begin{aligned} \partial_t u &= DT \cdot X \\ &= \psi(t)\phi(x)\partial_x u \qquad \text{by (2)} \\ &= \psi(t)\phi(x)T \cdot DX \\ &\text{or} \\ \frac{DT}{\psi(t)T(t)} &= \phi(x)\frac{DX}{X} \end{aligned}$$

Now, the only way that a smooth function of t can equal a function of x for all t and x in some open set is that both must equal the same *constant*

We therefore conclude that there is a constant C such that

$$\frac{DT}{\psi(t)T(t)} = C = \phi(x)\frac{DX}{X}$$

This reduces the problem to solving two ODE's, one for $T(t)$ and another for $X(x)$ and getting

$$u(x,t) = T(t) \cdot X(x)$$

Example (Heat Equation)

We consider the transfer of heat in a thin wire of length L. The heat flow at time t and position x is related to the change in temperature of position x at time t.

We assume the wire has coordinates $0 \leq x \leq L$ on the real line, and we let $u(x,t)$ denote the temperature at position x and time t.

The laws of heat conduction in this physical system can be used to derive the following partial differential equation for $u(x,t)$.

$$u_t = \alpha^2 u_{xx} \quad \forall t. \tag{3}$$

The constant α^2 depends on the conductive properties of the wire. Thus, for instance, it is different for copper or aluminum wires.

In general, we are interested in finding all solutions of (3). For mathematical convenience, we will impose other conditions to solve this problem. Thus, we assume that there is an initial temperature distribution $u(x,0) = f(x)$ in the wire and that the boundary points are kept at constant temperatures. This means that $u(0,t) = T_1$ and $u(L,t) = T_2$ where T_1 and T_2 are constants. Physically, the latter condition means that the ends of the wire are perfectly insulated, so that no heat flows in them.

General remarks.

1. (Principle of Superposition) If $u(x,t)$ and $v(x,t)$ are solutions to (3), and c_1, c_2 are constants, then $z(x,t) = c_1 u(x,t) + c_2 v(x,t)$ is also a solution to (3).

   Proof.

   We have

$$\begin{aligned}
\alpha^2 z_{xx} &= \alpha^2 (c_1 u_{xx} + c_2 v_{xx}) \\
&= c_1 \alpha^2 u_{xx} + c_2 \alpha^2 v_{xx} \\
&= c_1 u_t + c_2 v_t \\
&= (c_1 u + c_2 v)_t \\
&= z_t.
\end{aligned}$$

QED.

2. Using Remark 1, we can reduce to the case in which the boundary constants are both 0. This is called *homogeneous boundary conditions.*

   Indeed, note that any linear time independent function $w(x,t) = ax+b$ is a solution to (3), so we simply choose $w(x,t)$ so that $w(0,t) = T_1$ and $w(L,t) = T_2$. That is, we take

   $$w(x,t) = T_1 + \frac{T_2 - T_1}{L}x.$$

   Then, $\bar{u} = u - w$ is a solution to (3) such that $\bar{u}(0,t) = \bar{u}(L,t) = 0$, and we get $u(x,t) = \bar{u}(x,t) + w(x,t)$.

We want to find all solutions $u(x,t)$ to the problem (3) satisfying

$$u(x,0) = f(x), \tag{4}$$

and

$$u(0,t) = u(L,t) = 0 \quad \forall t. \tag{5}$$

Clearly the function $u(x,t) = 0$ is a solution, so we will only consider non-trivial solutions: $u(x,t) \neq 0$.

The PDE (3) is clearly separable, so let us apply the method of separation of variables. We try to find solutions $u(x,t)$ which decompose as a product of a function of x and one of t.

That is,

$$u(x,t) = X(x)T(t).$$

We get

$$\alpha^2 X''T = XT'$$

$$\alpha^2 \frac{X''}{X} = \frac{T'}{T}.$$

Since X only depends on x and T only depends on t, we must have that there is a constant β such that

$$\alpha^2 \frac{X''}{X} = \beta, \text{ and } \frac{T'}{T} = \beta.$$

This gives the two ordinary differential equations

$$X'' - \frac{\beta}{\alpha^2} X = 0, \tag{6}$$

and

$$T' = \beta T. \tag{7}$$

The last equation is easily solved

$$T(t) = T(0)e^{\beta t}.$$

Claim 1: The homogeneous boundary conditions imply that $\beta < 0$.

Proof of Claim 1: If $\beta > 0$, then the second equation has the form

$$X'' - \lambda X = 0$$

where $\lambda > 0$.

We may assume that $T(0) \neq 0$ since we are assuming $u(x,t)$ is not the trivial solution.

The general solution is

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Using $u(0,t) = 0$ we get $X(0) = 0$ or

$$c_1 + c_2 = 0.$$

Using $u(L,t) = 0$ we get $X(L) = 0$, or

$$c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L} = 0$$

This gives

$$c_1(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}) = 0.$$

If $c_1 \neq 0$, this gives

$$e^{\sqrt{\lambda}L} = e^{-\sqrt{\lambda}L},$$

Since $L \neq 0$, the first number above is greater than 1, but the second number is less than 1. Thus, $c_1 = c_2 = 0$. This contradiction proves the claim.

Now that we know $\beta < 0$, we write it as $-\sigma^2$ where $\sigma > 0$.

Claim 2: The homogeneous boundary conditions imply that σ must have the form

$$\sigma = \sigma_n = \frac{\alpha n\pi}{L}.$$

Proof of Claim 2:
We have the equation

$$X'' + \frac{\sigma^2}{\alpha^2}X = 0$$

The general solution to the first equation is

$$X(x) = c_1 cos(\frac{\sigma}{\alpha}x) + c_2 sin(\frac{\sigma}{\alpha}x).$$

Using $X(0) = 0$ we get $c_1 = 0$.
Using $X(L) = 0$, and $c_2 \neq 0$, we get

$$sin(\frac{\sigma}{\alpha}L) = 0,$$

or

$$\frac{\sigma}{\alpha}L = n\pi \text{ for some integer } n.$$

QED.

The considerations we have done so far give us that we can find solutions to (3) with homogeneous boundary conditions of the form

$$u_n(x,t) = T(0)e^{-\sigma_n^2 t}c_1 \sin(\frac{n\pi}{L}x),$$

where

$$\sigma_n = (\frac{\alpha n\pi}{L}).$$

Setting $T(0)c_1 = 1$, we get

$$u_n(x,t) = e^{-\sigma_n^2 t}\sin(\frac{n\pi}{L}x),$$

These are called fundamental solutions to the heat equation with homogeneous boundary conditions.

By superposition, we can also get solutions of the form

$$u(x,t) = \sum_{n=1}^{m} c_n u_n(x,t).$$

for a finite integer m. It turns out that if the series

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

actually converges, then it also represents a solution.

Next, considering the effect of the initial condition $u(x,0) = f(x)$ on this kind of solution, we get

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x,0) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L}x). \tag{8}$$

This is just a Fourier Sine series. It represents a function $F(x)$ defined on the whole line $\mathbf{R}$ which satifies

1. $F(x) = f(x)$ for $x \in [0,L]$,
2. $F(-x) = F(x)$ for all x. That is, $F(x)$ is an *odd* function.
3. $F(x + 2L) = F(x)$. That is, F is periodic of period $2L$.

Thus, we can take our original function $u(x,0) = f(x)$ defined only on $[0,L]$ and take its odd extension $F(x)$ of period $2L$ defined on all of $\mathbf{R}$.

Expanding $F(x)$ in a Fourier series we get

$$F(x) = \sum_{n=1}^{\infty} c_n u_n(x,0) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L}x). \tag{9}$$

and we can determine

$$c_n = \frac{1}{L}\int_{-L}^{L} F(x)sin(\frac{n\pi}{L}x)dx = \frac{2}{L}\int_{0}^{L} f(x)sin(\frac{n\pi}{L}x)dx. \tag{10}$$

These considerations enable one to solve the Homogeneous Heat Equation $u_t = \alpha^2 u_{xx}$ for various initial data $u(x,0) = f(x)$ on $0 \le x \le L$.

We proceed as follows:

Step 1: Expand f as a Fourier sine series on $[0, L]$.

More precisely, take the Fourier series of the odd extension of f to $[-L, L]$ (i.e., the periodic function of period $2L$ on $[-L, L]$ which agrees of f on $[0, L]$).

Thus, write

$$f(x) \sim \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}), \tag{11}$$

where

$$c_n = \frac{2}{L}\int_{0}^{L} f(x)sin(\frac{n\pi x}{L})dx. \tag{12}$$

Step 2: Insert the *multipliers*

$$e^{-t\sigma_n^2} = e^{-t(\frac{n\pi\alpha}{L})^2} \tag{13}$$

in each term of the sum to get

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-t\sigma_n^2} \sin(\frac{n\pi x}{L}), \tag{14}$$

or

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-t(\frac{n\pi\alpha}{L})^2} \sin(\frac{n\pi x}{L}). \tag{15}$$

Example: Solve the heat equation

$$u_t = 4u_x x, \quad u(x,0) = 3sin(\pi x) - 4sin(10\pi x), \ 0 \le x \le 5$$

Solution:

Here $\alpha = 2$.

We first find the Fourier sine series for $f(x) = u(x,0)$ on $[0,5]$.

This is

$$f(x) \sim \sum_{n\geq 1} c_n sin(\frac{n\pi x}{5})$$

But,

$$f(x) = 3sin(\pi x) - 4sin(10\pi x) = 3sin(\frac{5\pi x}{5}) - 4sin(\frac{10\cdot 5\pi x}{5})$$

is already a Fourier sine series. In fact, it is a *finite* one.

By uniqueness of Fourier series, our initial function is already the Fourier sine series we want.

Now we just add the multipliers to get

$$u(x,t) = 3e^{-t(\frac{5\pi 2}{5})^2} sin(\frac{5\pi x}{5}) - 4e^{-t(\frac{10\cdot 5\pi x}{5})^2} sin(\frac{10\cdot 5\pi x}{5})$$

or

$$u(x,t) = 3e^{-t\pi^2 4} sin(\pi x) - 4e^{-t(100\pi^2 4)} sin(10\pi x)$$

Note: As in the example, if the given function $u(x,0)$ is a sine series of the form

$$u(x,0) = \sum_{m\geq 1} A_m sin(m\pi x)$$

and the number L is a positive integer, then one can write $m = \frac{n\cdot L}{L}$ where $n\cdot L$ is also a positive integer, and one can obtain $u(t,x)$ as follows.

We replace each term $sin(m\pi x)$ in the sine series by $e^{-t(m\pi\alpha)^2} sin(m\pi x)$.

This works because, in writing the function $u(x,0)$ as a Fourier series, and inserting the exponential multipliers, we have to replace

$$sin(m\pi x)$$

by

$$e^{-t(\frac{n\cdot L\pi\alpha}{L})^2} sin(\frac{n\cdot L\pi x}{L})$$

and the $L's$ simply cancel.