22. Periodic Functions and Fourier Series

1 Periodic Functions

A real-valued function \( f(x) \) of a real variable is called periodic of period \( T > 0 \) if \( f(x + T) = f(x) \) for all \( x \in \mathbb{R} \).

For instance the functions \( \sin(x), \cos(x) \) are periodic of period \( 2\pi \). It is also periodic of period \( 2n\pi \), for any positive integer \( n \). So, there may be infinitely many periods. If needed we may specify the least period as the number \( T > 0 \) such that \( f(x + T) = f(x) \) for all \( x \), but \( f(x + s) \neq f(x) \) for \( 0 < s < T \).

For later convenience, let us consider piecewise \( C^1 \) functions \( f(x) \) which are periodic of period \( 2L > 0 \) where \( L \) is a positive real number. Denote this class of functions by \( \text{Per}_L(\mathbb{R}) \).

Note that for each integer \( n \), the functions \( \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \) are in examples of such functions. Also, note that if \( f(x), g(x) \in \text{Per}_L(\mathbb{R}) \), and \( \alpha, \beta \) are constants, then \( \alpha f + \beta g \) is also in \( \text{Per}_L(\mathbb{R}) \).

In particular, any finite sum

\[
\frac{a_0}{2} + \sum_{m=1}^{k} \left( a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)
\]

is in \( \text{Per}_L(\mathbb{R}) \). Here the numbers \( a_0, a_m, b_m \) are constants.
2 Fourier Series

The next result shows that in many cases the infinite sum

\[ f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos\left( \frac{m\pi x}{L} \right) + b_m \sin\left( \frac{m\pi x}{L} \right) \right) \]  (1)

determines a well-defined function \( f(x) \) which again is in \( Per_L(\mathbb{R}) \).

An infinite sum as in formula (1) is called a Fourier series (after the French engineer Fourier who first considered properties of these series).

**Fourier Convergence Theorem.** Let \( f(x) \) be a piecewise \( C^1 \) function in \( Per_L(\mathbb{R}) \). Then, there are constants \( a_0, a_m, b_m \) (uniquely defined by \( f \)) such that at each point of continuity of \( f(x) \) the expression on the right side of (1) converges to \( f(x) \). At the points \( y \) of discontinuity of \( f(x) \), the series converges to

\[ \frac{1}{2}(f(y^-) + f(y^+)). \]

The values \( f(y^-), f(y^+) \) denote the left and right limits of \( f \) as \( x \to y \), respectively.

That is,

\[ f(y^-) = \lim_{x \to y, x < y} f(x), \quad f(y^+) = \lim_{x \to y, x > y} f(x). \]
It turns out that the constants $a_0, a_m, b_m$ above are determined by the formulas

\[ a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx \]  
(2)

\[ a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) \, dx \]  
and

\[ b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) \, dx. \]  
(4)

We will justify this a bit later, but for now, let us use these formulas to compute some Fourier series. The constants $a_-, a_m, b_m$ are called the Fourier coefficients of $f$.

**Example 1.** Let $f(x)$ be defined by

\[ f(x) = \begin{cases} 
-x, & -2 \leq x < 0 \\
x, & 0 \leq x < 2
\end{cases} \]

and $f(x + 4) = f(x)$ for all $x$.

Determine the Fourier coefficients of $f$.

Note that the graph of this function $f(x)$ looks like a “triangular wave.”

Here $L = 2$, and we compute

\[ a_0 = \frac{1}{2} \int_{-2}^{0} (-x) \, dx + \frac{1}{2} \int_{0}^{2} x \, dx \]
\[ = 1 + 1 = 2, \]

\[ a_m = \frac{1}{2} \int_{-2}^{2} f(x) \, dx \]
and, for $m > 0$,

\[
a_m = \frac{1}{2} \int_{-2}^{0} (-x) \cos\left(\frac{m\pi x}{2}\right) dx + \frac{1}{2} \int_{0}^{2} x \cos\left(\frac{m\pi x}{2}\right) dx + \frac{1}{2} \int_{0}^{2} x \sin\left(\frac{m\pi x}{2}\right) dx.
\]

\[
b_m = \frac{1}{2} \int_{-2}^{0} (-x) \sin\left(\frac{m\pi x}{2}\right) dx + \frac{1}{2} \int_{0}^{2} x \sin\left(\frac{m\pi x}{2}\right) dx.
\]

To compute these integrals, we note that, integration by parts gives the formulas

\[
\int x \cos(ax) dx = \frac{x}{a} \sin(ax) - \int \frac{\sin(ax)}{a} dx
\]

\[
= \frac{x}{a} \sin(ax) + \frac{\cos(ax)}{a^2}
\]

(5)

\[
\int x \sin(ax) dx = -\frac{x}{a} \cos(ax) + \frac{\sin(ax)}{a^2}
\]

Let us compute $a_m$.

First note that the integrand

\[h(x) = f(x) \cos\left(\frac{m\pi x}{2}\right)\]

satisfies $h(-x) = h(x)$. Hence, the two integrals are equal and

\[a_m = 2(1/2) \int_{0}^{2} x \cos\left(\frac{m\pi x}{2}\right) dx = \int_{0}^{2} x \cos\left(\frac{m\pi x}{2}\right) dx\]

Using the formula for (5) above, we get
\[ a_m = \int_0^2 x \cos\left(\frac{m\pi x}{2}\right)dx \]
\[ = \left[\left(\frac{2}{m\pi}\right)x \sin\left(\frac{m\pi x}{2}\right) + \left(\frac{2}{m\pi}\right)^2 \cos\left(\frac{m\pi x}{2}\right)\right]_0^2 \]
\[ = 0 + \left(\frac{4}{m^2\pi^2}\right)(\cos(m\pi) - 1) \]
\[ = \begin{cases} 
-\frac{8}{(m\pi)^2}, & m \text{ odd} \\
0, & m \text{ even} 
\end{cases} \]

A similar calculation shows that \( b_m = 0 \) for all \( m \). We will see later that this last fact follows from the fact that \( f(-x) = f(x) \) for all \( x \).

**Example 2.** Let \( f(x) \) be defined by
\[ f(x) = \begin{cases} 
0, & -3 < x < -1, \\
1, & -1 < x < 1, \\
0, & 1 < x < 3 
\end{cases} \]
\[ f(x + 6) = f(x) \quad \text{for all} \quad x. \]
Find the Fourier series for \( f \).

Here \( L = 3 \), and we compute
\[ a_0 = \frac{1}{3} \int_{-3}^3 f(x)\,dx \]
\[ = \frac{1}{3} \int_{-1}^1 f(x)\,dx \]
For \( m > 0 \), we have

\[
a_n = \frac{1}{3} \int_{-1}^{1} \cos \left( \frac{n\pi x}{3} \right) dx
= \left[ \frac{1}{n\pi} \sin \left( \frac{n\pi x}{3} \right) \right]_{-1}^{1}
= \frac{2}{n\pi} \sin \left( \frac{n\pi}{3} \right)
\]

\[
b_n = \frac{1}{3} \int_{-1}^{1} \sin \left( \frac{n\pi x}{3} \right) dx
= -\left[ \frac{1}{n\pi} \cos \left( \frac{n\pi x}{3} \right) \right]_{-1}^{1}
= -\frac{1}{n\pi} \left[ \cos \left( \frac{n\pi}{3} \right) - \cos \left( \frac{-n\pi}{3} \right) \right]
= 0
\]

3 Justification of the Fourier coefficient formulas

We need the following basic facts about the integrals of certain products of sines and cosines.
Note: The handwritten pages below are to provide motivation for the calculation of Fourier coefficients.

The notes in this section can be read by skipping the handwritten additions on this and the next four pages.
linear space of functions

Set \( L \) of functions which
is closed under + and scalar multiplication

\( \mathbf{\oplus} \) \( f, g \in L \Rightarrow f + g \in L \)

\( \mathbf{\circ} \) \( f, g \in L \Rightarrow a \cdot f \in L \)

E.g. 1) Cont functions = \( L \)

2) Diff functions = \( L \)

3) \( \mathbb{R}^n \)

An inner prod \( \langle \cdot, \cdot \rangle \) on \( L \) is

a map \( \langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{R} \)

s.t. (1) \( \langle f, g \rangle = \langle g, f \rangle \) – symmetry

(2) \( \langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle \)

(3) \( \langle f, f \rangle \geq 0 \) and = 0 iff \( f = 0 \)

define \( ||f|| = \sqrt{\langle f, f \rangle} \) – norm prod.

\( f \) is a unit vector in \( L \)

\[ ||f|| = 1 \]

\( f \) is orthogonal to \( g \) if \( \langle f, g \rangle = 0 \)
$L = \text{Cont periodic fns on } [-L,L]$ 

Define for $f, g \in L$

set $<f, g> = \frac{1}{L} \int_{-L}^{L} f(x) \cdot g(x) \, dx$

$f \perp g \iff <f, g> = 0$

**Facts:**

$<\cos \left(\frac{m \pi x}{L}\right), \cos \left(\frac{n \pi x}{L}\right)> = 0 \quad \forall m \neq n$

$<\cos \left(\frac{m \pi x}{L}\right), \sin \left(\frac{n \pi x}{L}\right)> = 0$

$<\sin \left(\frac{m \pi x}{L}\right), \sin \left(\frac{n \pi x}{L}\right)> = \delta_{mn}$ if $m \neq n$

$<\cos \left(\frac{m \pi x}{L}\right), \cos \left(\frac{m \pi x}{L}\right)> = 1 \quad \text{if } m = n$

$\sin \left(\frac{m \pi x}{L}\right) \sin \left(\frac{n \pi x}{L}\right) = \delta_{mn}$ if $m \neq 0$

An orthonormal set in $L$ is a set $\{a_n\}$ s.t.

$<a_n, a_m> = \delta_{mn}$

$= 1 \quad \text{if } m = n$

$= 0 \quad \text{if } m \neq n$

Now suppose an $f \in L$ can be written as $f = \sum_{j=1}^{k} \alpha_j f_j$ where

$\{f_1, \ldots, f_k\}$ is a finite orthonormal set of functions in $L$

How can we get the coefficients $\alpha_i$?
The answer is to compute the inner product \( \langle f, f_i \rangle \).

By linearity we have
\[
\langle f, f_i \rangle = \langle \sum_{j=1}^{k} \alpha_j f_j, f_i \rangle
\]
\[
= \sum_{j=1}^{k} \alpha_j \langle f_j, f_i \rangle
\]
\[
= \sum_{j=1}^{k} \alpha_j \delta_{ji} = \alpha_i \quad \text{since} \quad \delta_{ji} = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}
\]

If we can extend this to infinite orthonormal sets \( \{ f_1, f_2, \ldots \} \), then the same argument gives
\[
\langle f, f_i \rangle = \langle \sum_{j=1}^{\infty} \alpha_j f_j, f_i \rangle = \alpha_i
\]

The formulas (6), (7), (8) below (or the handwritten facts above) can be interpreted as saying that the set of functions
\[
\mathcal{F} = \{ \cos \left( \frac{m \pi x}{L} \right), \sin \left( \frac{n \pi x}{L} \right), \quad n=1,2,\ldots, \quad m=1,2,3,\ldots \}
\]
is orthonormal in \( P_{\infty}(R) \).
Consider the Fourier Series for
\[ f = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \left( \frac{m \pi x}{L} \right) + b_m \sin \left( \frac{m \pi x}{L} \right) \right) \]

So,
\[ f - \frac{a_0}{2} = \sum_{m=1}^{\infty} \left( a_m \cos \left( \frac{m \pi x}{L} \right) + b_m \sin \left( \frac{m \pi x}{L} \right) \right) \]

and \( \langle f - \frac{a_0}{2}, \cos \left( \frac{m \pi x}{L} \right) \rangle = a_m \) (Since \( f_0 \) is orthonormal)

but also \( \langle f, \cos \left( \frac{m \pi x}{L} \right) \rangle - \langle \frac{a_0}{2}, \cos \left( \frac{m \pi x}{L} \right) \rangle \)

\[ \text{0 since } \frac{1}{L} \int_{L}^{0} \frac{a_0}{2} \cos \left( \frac{m \pi x}{L} \right) \]

and we get
\[ \langle f, \cos \left( \frac{m \pi x}{L} \right) \rangle = a_m \]

Similarly, \( \langle f, \sin \left( \frac{m \pi x}{L} \right) \rangle = b_m \)
\[ \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \neq 0, \\ L, & m = n = 0 \end{cases} \] (6)

\[ \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0 \text{ for all } m, n; \] (7)

\[ \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \neq 0, \\ 0, & m = n = 0 \end{cases} \] (8)

We justify formula (8), leaving the other similar calculations to the reader. First recall some formulas related to the sine and cosine functions.

The sum and difference formulas are:

\[ \cos(\alpha + \beta) = \cos(\alpha)(\cos(\beta) - \sin(\alpha) \sin(\beta)) \] (9)

\[ \cos(\alpha - \beta) = \cos(\alpha)(\cos(\beta) + \sin(\alpha) \sin(\beta)). \] (10)

Applying the first formula with \( \alpha = \beta \) gives

\[ \cos(2\alpha) = \cos(\alpha)^2 - \sin(\alpha)^2 \]
This implies that

\[ 1 + \cos(2\alpha) = \cos(\alpha)^2 + \sin(\alpha)^2 + \cos(\alpha)^2 - \sin(\alpha)^2 \]
\[ = 2 \cos(\alpha)^2 \]

or the so-called \textit{cosine half-angle formula}

\[ \cos(\alpha)^2 = \frac{1}{2}(1 + \cos(2\alpha)). \]

Similarly, the \textit{sine half-angle formula} is

\[ \sin(\alpha)^2 = \frac{1}{2}(1 - \cos(2\alpha)). \]

Formulas (9) and (10) imply that

\[ \cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin(\alpha) \sin(\beta). \]

Using \( \alpha = nx, \ \beta = mx \) then gives

\[ \cos((n - m)x) - \cos((n + m)x) = 2 \sin(nx) \sin(mx). \]  \hspace{1cm} (11)

This implies, for \( m \neq n \) and both positive,

\[
\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^{L} \cos\left(\frac{(m - n)\pi x}{L}\right) dx \\
- \frac{1}{2} \int_{-L}^{L} \cos\left(\frac{(m + n)\pi x}{L}\right) dx
\]
\[ \frac{L}{2\pi} \left[ \frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{m - n} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{m + n} \right]_{-L}^L = 0. \]

If \( m = n = 0 \), then
\[ \int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \int_{-L}^L 0 \, dx = 0, \]
while if \( m = n \neq 0 \), we have
\[ \int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \int_{-L}^L \left(\sin\left(\frac{m\pi x}{L}\right)\right)^2 \, dx = \frac{1}{2} \int_{-L}^L \left[ 1 - \cos\left(\frac{2m\pi x}{L}\right) \right] \, dx = \frac{1}{2} \left[ x - \sin\left(\frac{2m\pi x}{2}\right) \right]_{-L}^L = L. \]

Now, suppose that
\[ f(x) = \frac{a_0}{2} + \sum_{m \geq 1} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right). \quad (12) \]

Since the integrals of cosine and sine functions over intervals of lengths equal to their periods vanish, we have
\[ \int_{-L}^L f(x) \, dx = \int_{-L}^L \left(\frac{a_0}{2} + \sum_{m \geq 1} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right) \, dx \]
\[
= \frac{a_0}{2} (2L) + \sum_{m \geq 1} \int_{-L}^{L} a_m \cos \left( \frac{m\pi x}{L} \right) dx \\
+ \sum_{m \geq 1} \int_{-L}^{L} b_m \sin \left( \frac{m\pi x}{L} \right) dx \\
= a_0 L
\]

Analogously, using the orthogonality relations above, we have that, for \( n \geq 1, \)
\[
\int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx = \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \left( \frac{a_0}{2} + \sum_{m \geq 1} a_m \cos \left( \frac{m\pi x}{L} \right) \right) dx \\
+ \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \left( \sum_{m \geq 1} b_m \sin \left( \frac{m\pi x}{L} \right) \right) dx \\
= a_m L
\]

which gives (3). Formula (4) is justified in a similar way.

4 Even and Odd functions

A function \( f(x) \) is called even if \( f(-x) = f(x) \) for all \( x \). Analogously, a function \( f(x) \) is called odd if \( f(-x) = -f(x) \) for all \( x \). For example, \( \cos(x) \) is even, and \( \sin(x) \) is odd.

Also, one sees easily that linear combinations of even (odd) functions are again even (odd).

The following facts are useful.
1. The product of two odd functions is even.

2. The product of two even functions is even.

3. The product of an even function and an odd function is odd.

The following comments are useful.

4. The Fourier series of a periodic odd function only involves sine terms: i.e.; \( a_m = 0 \) for all \( m \geq 1 \). This is because

\[
a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx
\]

and the function \( f(x) \cos\left(\frac{m\pi x}{L}\right) \) is odd.

5. Similarly, the Fourier series of a periodic even function only involves cosine terms: i.e.; \( b_m = 0 \) for all \( m \geq 1 \).

6. Any function \( f(x) \) defined on \([0, L]\) (with \( L > 0 \) has an even extension \( f_{\text{even}}(x) \) to \([-L, L]\) and also has an odd extension \( f_{\text{odd}}(x) \) to \([-L, L]\).

The definitions are

\[
f_{\text{even}}(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x \leq 0 \end{cases}
\]

\[
f_{\text{odd}}(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ -f(-x) & -L \leq x \leq 0 \end{cases}
\]
Hence, any piecewise continuous function $f(x)$ on $[0, L]$ can be represented both as Fourier cosine series and a Fourier sine series.