21a. Some examples of heat equation problems

Consider the heat equation

\[ \alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 \]
\[ u(0, t) = 0 = u(1, t), \quad t > 0 \]
\[ u(x, 0) = f(x), \quad 0 \leq x \leq L. \]

We have that the solution is

\[ u(x, t) = \sum_{n=1}^{\infty} c_n \exp \left( -\frac{n^2 \pi^2 \alpha^2 t}{L^2} \right) \sin \left( \frac{n \pi x}{L} \right) \quad (1) \]

where

\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right), \quad (2) \]

and

\[ c_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) dx. \quad (3) \]

Remark. Note that from formula (3) the coefficients \( c_n \) are uniquely determined by \( f \) and \( L \).

Let us use this to solve the following heat equation problems.

1. 50 \( u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0 \)
\[ u(0, t) = 0 = u(1, t), \quad t > 0 \]
\[ u(x, 0) = 2 \sin(2\pi x) + 6 \sin(8\pi x), \quad 0 \leq x \leq 1. \]
Solution: In this case, $\alpha^2 = 50$, $L = 1$.

From (1) we have

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp\left(-n^2\pi^2 50t\right) \sin(n\pi x).$$

We are given $f(x)$ as a sum of terms $c_n \sin(n\pi x)$ with known $c_n$. By the uniqueness of the $c'_n$s, the only non-zero terms are for $n = 2$ and $n = 8$, and we must have that $c_2 = 2, c_8 = 6$.

So the answer is

$$u(x, t) = 2 \exp(-200\pi^2 t) \sin(2\pi x) + 6 \exp(-64 \cdot 50\pi^2 t) \sin(8\pi x).$$

$$u(x, t) = 2 e^{-200\pi^2 t} \sin(2\pi x) + 6 e^{-64 \cdot 50\pi^2 t} \sin(8\pi x).$$

2. $100 u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0$

$u(0, t) = 0 = u(1, t), \quad t > 0$

$u(x, 0) = 2 \sin(2\pi x) + 6 \sin(3\pi x), \quad 0 \leq x \leq 4.$

Solution: Here $\alpha^2 = 100$ and $L = 4$.

So the solution has the form
\[ u(x, t) = \sum_{n=1}^{\infty} c_n \exp \left( -\frac{n^2 \pi^2 100 t}{16} \right) \sin \left( \frac{n \pi x}{4} \right). \]

The terms \(2\pi x\) and \(3\pi x\) inside the initial condition
\[ u(x, 0) = 2 \sin(2\pi x) + 6 \sin(3\pi x) \]

have the form
\[ n\pi x/4 \]

with \(n = 8\) and \(n = 12\), respectively. So the only \(n\)'s with \(c_n\) not zero are \(n = 8\) and \(n = 12\). We also have \(c_8 = 2\), \(c_{12} = 6\).

So the answer is
\[
\begin{align*}
\quad u(x, t) & = 2 \exp \left( -\frac{64 \pi^2 100 t}{16} \right) \sin \left( \frac{8 \pi x}{4} \right) \\
& \quad + 6 \exp \left( -\frac{144 \pi^2 100 t}{16} \right) \sin \left( \frac{12 \pi x}{4} \right) \\
& = 2 \exp \left( -\frac{64 \pi^2 100 t}{16} \right) \sin(2\pi x) \\
& \quad + 6 \exp \left( -\frac{144 \pi^2 100 t}{16} \right) \sin(3\pi x).
\end{align*}
\]

See section 23 for more examples of problems.