Heat equation \[ u_t = \alpha^2 u_{xx} \]

Look for solutions \( u(x,t) \) on \([0, L] \)

\[
\begin{align*}
    u(0,t) &= 0, \quad u(L,t) = 0 \quad \forall t \geq 0 \\
    u(x,0) &= f(x), \qquad \text{initial temp distribution}
\end{align*}
\]

Try to find solutions

\[ u(x,t) = T(t) \overline{X}(x) \]

Found only solutions had form

For \( n \geq 0 \), get \( \overline{X}_n(x) = c_n \sin \left( \frac{n \pi x}{L} \right) \)

\[
T_n(t) = e^{-\left(\frac{n \pi \alpha}{L}\right)^2 t}
\]

Try \( u(x,t) = T(t) \overline{X}(x) \)

\[
\begin{align*}
    u_t &= T' \overline{X}, \quad u_{xx} = T \overline{X}'' \\
    T' \overline{X} &= \alpha^2 T \overline{X}'' \\
    \alpha^2 \frac{\overline{X}''}{\overline{X}} &= \frac{T'}{T} = -\sigma
\end{align*}
\]
Get \( \frac{d^2}{dx^2} \overline{X}'' = -\sigma \frac{d^2}{dx^2} \overline{X} + \sigma \overline{X} = 0 \)
\( \overline{X}'' + \frac{\sigma}{\alpha^2} \overline{X} = 0 \)
\[
I^1 \quad \frac{T}{\alpha} = -\sigma, \quad T^1 = -\sigma T \quad | \quad \lambda^2 = \frac{\sigma}{\alpha^2}
\]
Solutions: \( \overline{X} = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \)
\( T = T/\alpha \) \( e^{-\sigma t} \)
\( u = T f(x) \overline{X}(t) \)
\[
\Rightarrow \quad e^{-\sigma t} \left( c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \right)
\]
\( t \to 0 \Rightarrow \)
+ Boundary conditions \( c_1 = 0 \)
\( e^{-\sigma t} c_2 \sin(\lambda t) \)
\( t = 0, \) \( u(x,0) = f(x) = c_2 \sin(\lambda x) \)
\( \sin(\lambda L) = 0 \quad \frac{\lambda L}{\pi} = n \quad n = 0, 2, \ldots \)
So, \( u_n(x,t) = c_n e^{-\sigma_n t} \sin \left( \frac{n\pi x}{L} \right) \)
So, any \( u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\sigma_n t} \sin \left( \frac{n\pi x}{L} \right) \)
\[ u(x,0) = f(x) = \frac{1}{L} \sum_{n=0}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right) \]

Also,

\[ f(x) = \sum_{n=0}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right) \]

is a Fourier sine series.

Any reasonable \( f(x) \) on \([0, L]\) s.t. \( f(0) = f(L) = 0 \) can be represented as

We consider the transfer of heat in a thin wire of length $L$. The heat flow at time $t$ and position $x$ is related to the change in temperature of position $x$ at time $t$.

We assume the wire has coordinates $0 \leq x \leq L$ on the real line, and we let $u(x, t)$ denote the temperature at position $x$ and time $t$.

The laws of heat conduction in this physical system can be used to derive the following partial differential equation for $u(x, t)$.

$$\alpha^2 u_{xx} = u_t \; \forall t. \tag{1}$$

The constant $\alpha^2$ depends on the conductive properties of the wire. Thus, for instance, it is different for copper or aluminum wires.

In general, we are interested in finding all solutions of (1). For mathematical convenience, we will impose other conditions to solve this problem. Thus, we assume that there is an initial temperature distribution $u(x, 0) = f(x)$ in the wire and that the boundary points are kept at constant temperatures. This means that $u(0, t) = T_1$ and $u(L, t) = T_2$ where $T_1$ and $T_2$ are constants. Physically, the latter condition means that the ends of the wire are
perfectly insulated, so that no heat flows in them.

**General remarks.**

1. (Principle of Superposition) If \( u(x, t) \) and \( v(x, t) \) are solutions to (1), and \( c_1, c_2 \) are constants, then \( z(x, t) = c_1 u(x, t) + c_2 v(x, t) \) is also a solution to (1).

**Proof.**

We have

\[
\alpha^2 z_{xx} = \alpha^2(c_1 u_{xx} + c_2 v_{xx})
\]

\[= c_1 \alpha^2 u_{xx} + c_2 \alpha^2 v_{xx} \]

\[= c_1 u_t + c_2 v_t \]

\[= (c_1 u + c_2 v)_t \]

\[= z_t. \]

QED.

2. Using Remark 1, we can reduce to the case in which the boundary constants are both 0. This is called **homogeneous boundary conditions.**

   Indeed, note that any linear time independent function \( w(x, t) = ax + b \) is a solution to (1), so we simply choose \( w(x, t) \) so that \( w(0, t) = T_1 \) and \( w(L, t) = T_2 \). That is, we take
\[ w(x, t) = T_1 + \frac{T_2 - T_1}{L} x. \]

Then, \( \bar{u} = u - w \) is a solution to (1) such that \( \bar{u}(0, t) = \bar{u}(L, t) = 0 \), and we get \( u(x, t) = \bar{u}(x, t) + w(x, t) \).

We want to find all solutions \( u(x, t) \) to the problem (1) satisfying

\[ u(x, 0) = f(x), \quad \text{(2)} \]

and

\[ u(0, t) = u(L, t) = 0 \quad \forall t. \quad \text{(3)} \]

Clearly the function \( u(x, t) = 0 \) is a solution, so we will only consider non-trivial solutions: \( u(x, t) \neq 0 \).

As a preliminary guess, let us try to find solutions \( u(x, t) \) which decompose as a product of a function of \( x \) and one of \( t \).

That is,

\[ u(x, t) = X(x)T(t). \]

This method is called the method of separation of variables.
We get
\[ \alpha^s X''T = XT' \]
\[ \alpha^2 \frac{X''}{X} = \frac{T'}{T}. \]

Since \( X \) only depends on \( x \) and \( T \) only depends on \( t \), we must have that there is a constant \( \beta \) such that
\[ \alpha^2 \frac{X''}{X} = \beta, \text{ and } \frac{T'}{T} = \beta. \]

This gives the two ordinary differential equations
\[ X'' - \frac{\beta}{\alpha^2} X = 0, \quad (4) \]
and
\[ T' = \beta T. \quad (5) \]

The last equation is easily solved
\[ T(t) = T(0)e^{\beta t}. \]

**Claim 1:** The homogeneous boundary conditions imply that \( \beta < 0. \)

**Claim 2:** The homogeneous boundary conditions imply that \( \sigma \) must have the form
\[ \sigma = \sigma_n = \frac{\alpha^2 n^2 \pi^2}{L^2}. \]
**Proof of Claim 1:** If $\beta > 0$, then the second equation has the form

$$X'' - \lambda X = 0$$

where $\lambda > 0$.

We may assume that $T(0) \neq 0$ since we are assuming $u(x, t)$ is not the trivial solution.

The general solution is

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Using $u(0, t) = 0$ we get $X(0) = 0$ or

$$c_1 + c_2 = 0.$$

Using $u(L, t) = 0$ we get $X(L) = 0$, or

$$c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L} = 0$$

This gives

$$c_1(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}) = 0.$$

If $c_1 \neq 0$, this gives

$$e^{\sqrt{\lambda}L} = e^{-\sqrt{\lambda}L},$$

Since $L \neq 0$, the first number above is greater than 1, but the second number is less than 1. Thus, $c_1 = c_2 = 0$. This contradiction proves the claim.
Now that we know $\beta < 0$, we write it as $-\sigma$ where $\sigma > 0$.

**Proof of Claim 2:**

We have the two equations

$$X'' + \frac{\sigma}{\alpha^2}X = 0, \quad T(t) = T(0)e^{-\sigma t}$$

The general solution to the first equation is

$$X(x) = c_1\cos\left(\frac{\sqrt{\sigma}}{\alpha}x\right) + c_2\sin\left(\frac{\sqrt{\sigma}}{\alpha}x\right).$$

Using $X(0) = 0$ we get $c_1 = 0$.

Using $X(L) = 0$, and $c_2 \neq 0$, we get

$$\sin\left(\frac{\sqrt{\sigma}}{\alpha}L\right) = 0,$$

or

$$\frac{\sqrt{\sigma}}{\alpha}L = n\pi \text{ for some integer } n.$$

QED.

The considerations we have done so far give us that we can find solutions to (1) with homogeneous boundary conditions of the form

$$u_n(x, t) = e^{-\sigma_n t} \sin\left(\frac{n\pi}{L}x\right), \quad \sigma_n = \frac{n^2\pi^2}{\alpha L}$$

where

$$T(t) \mathcal{X}(x).$$
\[ \sigma_n = \left( \frac{\alpha n \pi}{L} \right)^2. \]

These are called fundamental solutions to the heat equation with homogeneous boundary conditions.

By superposition, we can also get solutions of the form

\[ u(x, t) = \sum_{n=1}^{m} c_n u_n(x, t). \]

for a finite integer \( m \). It turns out that if the series

\[ u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) \]

actually converges, then it also represents a solution.

Next, considering the effect of the initial condition \( u(x, 0) = f(x) \) on this kind of solution, we get

\[ f(x) = \sum_{n=1}^{\infty} c_n u_n(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left( \frac{n\pi}{L} x \right), \quad (6) \]

and we are led to consider what functions \( f(x) \) satisfy this last condition with a convergent series.

Note that such a function satisfies the following two properties.

1. \( f(x) \) is defined on the whole line \( \mathbb{R} \),
2. \( f(-x) = f(x) \) for all \( x \). That is, \( f(x) \) is an odd function.
3. \( f(x + 2L) = f(x) \). That is, \( f \) is periodic of period \( 2L \).

It is an amazing fact (which we will discuss in the next section) that every piecewise \( C^1 \) odd function \( f(x) \) of period \( 2L \) can be written in the form (6). The series converges to \( f(x) \) at all points at which \( f(x) \) is continuous.

Our original initial function \( f(x) = u(x, 0) \) was only defined in the interval \([0, L]\). We can extend this function \( f(x) \) to \([-L, L]\) by requiring that \( f(-x) = -f(x) \) for all \( x \). We will see in the next section that the coefficients \( c_n \) can be computed from the formula

\[
c_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx. \tag{7}
\]

Note that since \( f(x) \) is odd, the integrand in the previous formula is even. So, the integral over \([-L, 0]\) equals the integral over \([0, L]\). Hence we also have the formula

\[
c_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx. \tag{8}
\]

These considerations enable one to solve the heat equation for various initial values \( u(x, 0) = f(x) \) on \( 0 \leq x \leq L \).