Matrix Methods for 2-dim systems of linear d.e.'s.

Consider the system
\[ \dot{x} = Ax, \quad x \in \mathbb{R}^n, \quad x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \quad \dot{x} = \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{pmatrix} \]

Try to find a solution form
\[ x(t) = e^{rt} v \quad \text{where} \quad v \neq 0. \]

Then,
\[ \dot{x} = re^{rt} v = Ax = Ae^{rt} v. \]

or \[ e^{rt} rv = Av. \quad (1) \]

A number \( r \) s.t. \( \text{Non-zero vector} \)
\( v \) sat.(1) is called an eigenvalue of \( A \).

Any non-zero \( v \) as in (1) is an associated eigenvector to \( r \).

We can write \( Av = rv \) as \( Av = rIv \)
\( I = \text{Ident matrix} \)

or \( (A - rv) = 0 \) or \( (rIA) = 0 \)
\( n \times n \text{ matrix} \)

So, need \( \det (rIA) = 0 \)

- characteristic polynomial
Reversing steps we get

1) If \( \lambda \) is an eigenvalue of \( A \) with associated eigenvector \( \mathbf{v} \) then \( x(t) = e^{\lambda t} \mathbf{v} \) is a solution to \( \dot{x} = Ax \).

2) Now suppose

2) A set \( \{y_1(t), \ldots, y_n(t)\} \) of \( n \)-vector valued functions on \( t \in \text{interval} \) \( J \) is real

is called linearly independent if

\[
\sum_{i=1}^{n} c_i y_i(t) = 0 \quad \text{for} \quad t \in J \Rightarrow c_i = 0 \quad \text{for all} \quad i
\]

3) If \( \{y_1(t), \ldots, y_n(t)\} \) is a linearly independent set of solutions to \( \dot{x} = Ax \), then it is called a fundamental set of solutions.

Once such a fundamental set is found, every solution to \( \dot{x}(t) \) to (1) can be expressed uniquely as

\[
x(t) = \sum_{i=1}^{n} c_i y_i(t). \quad \text{General solution to (1)}
\]

Here the \( c_i \) depend on initial conditions.
Let $y_1(t), \ldots, y_n(t)$ be $n$ solutions to (1).

Form the matrix whose column vectors are the $y_i(t)$'s.

The Wronskian det of the set $y_1(t), \ldots, y_n(t)$ is

$$\text{det}(\mathbf{y}(t)) = W(y_1(t), \ldots, y_n(t))$$

is a real valued function of $t$.

The set $y_1(t), \ldots, y_n(t)$ is a fundamental set if $W(y_1(t), \ldots, y_n(t)) \neq 0$ for some $(t_0, y_{t_0})$.

Now we consider the 2-dim case.

So we consider

$x = a_{11} x + a_{12} y$

$y = a_{21} x + a_{22} y$

In vector notation, we have

$x = \begin{pmatrix} x \\ y \end{pmatrix}$, \quad $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$x = A x$.

The characteristic 
Want to find 2 lin indep solns.
Step 1: Find the characteristic polynomial

\[ z(r) \equiv (a_{11} + a_{12})r + a_{11}a_{22} - a_{21}a_{12} \]
\[ r^2 = \text{tr}(A) - r + \det(A). \]

Step 2: Factor \( z(r) \).

Case 1: \( z(r) = (r-r_1)(r-r_2) \), \( r_1 \neq r_2 \) real

Find associated eigenvectors.

\[(A-rI)\mathbf{v} = 0 \quad \text{let} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]
\[
\begin{pmatrix}
  a_{11} - r & a_{12} \\
  a_{21} & a_{22} - r
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix} = \begin{pmatrix} 0 \\
  0 \end{pmatrix}
\]

Since \( \det(A-rI) = 0 \) for \( r = r_1 \) or \( r_2 \), these equations are multiples of each other.

So, enough to look at \( (a_{11} - r)v_1 + a_{12}v_2 = 0 \) (Assume \( a_{22} \neq 0 \)).

Can take \( v_1 = 1 \),
\[ v_2 = \frac{r-a_{11}}{a_{12}} \quad \text{or} \quad \mathbf{v} = \begin{pmatrix} 1 \\ \frac{r-a_{11}}{a_{12}} \end{pmatrix}. \]
Plug in \( r = r_1 \), \( r = r_2 \) and get
\[
\begin{pmatrix} V \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{r_1-a_{11}}{a_{12}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} W \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{r_2-a_{11}}{a_{12}} \\ 1 \end{pmatrix}
\]
associate to \( r_1 \), associate to \( r_2 \).

Then, general solution to \( \dot{x} = Ax \)
\[
x(t) = \begin{pmatrix} \frac{r_1}{r_1t} \\ e^{r_1t} \end{pmatrix} + \begin{pmatrix} \frac{r_2}{r_2t} \\ e^{r_2t} \end{pmatrix}
\]
where \( c_1, c_2 \) are constants.

Consider IVP: \( \dot{x} = Ax, \ x(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \)

We then have
\[
x(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} C_1V + C_2W \\ V_1 \end{pmatrix} = \begin{pmatrix} V_1 & W_1 \\ V_2 & W_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}
\]
\[
or \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} V_1 & W_1 \\ V_2 & W_2 \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
\]

Note: If \( a_{12} = 0 \) but \( a_{21} \neq 0 \) can use 2nd equation in \((A - I)v = 0\)
or \( a_{21}V_1 + (a_{22} - r)V_2 = 0, \ V_2 = 0, \ V_1 = \frac{r - a_{22}}{a_{21}} \).
Note: If $v = (v_1, v_2)$ is an eigenvector for $A$, any multiple of $v$ is also an eigenvector.

So, we can get a unit eigenvector by taking

$$u = \frac{v}{\|v\|} = \left( \frac{v_1}{\sqrt{v_1^2 + v_2^2}}, \frac{v_2}{\sqrt{v_1^2 + v_2^2}} \right)$$
Case 2: \( z(r) = (r-r^2)^2 \)

Get eigenvector \( \mathbf{v} \) as before

and one soln \( x_1(t) = e^{rit} \mathbf{v} \).

Case 2a: There are two independent eigenvectors, say another one is \( \mathbf{w} \).

\( e.g. x = 2x \)

\( y = 2y \) Then, gen soln is

\[ x(t) = c_1 e^{rit} \mathbf{v} + c_2 e^{rit} \mathbf{w} \]

Case 2b: Only have multiples of \( \mathbf{v} \) as eigenvector

Find out from solutions of \( (A-r_1 I) \mathbf{v} = 0 \)

Then, get a 2nd independent solution to \( \dot{x} = Ax \)

of the form \( x_2(t) = e^{rit} \mathbf{w} + te^{rit} \mathbf{v} \)

where \( (A-r_1 I) \mathbf{w} = \mathbf{v} \).

So, general soln

\[ x(t) = c_1 e^{rit} \mathbf{v} + c_2 (e^{rit} \mathbf{w} + te^{rit} \mathbf{v}) \]