15. Step Functions and initial value problems with discontinuous forcing

In applications it is frequently useful to consider differential equations whose forcing terms are piecewise differentiable.

Here we begin to explore techniques which enable us to deal with this situation.

Def. Let $c > 0$ be a positive real number. The *Heaviside c* function is the function u_c defined by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

There are some simple piecewise continuous functions which can be constructed using simple operations applied to various $u_c's$.

Example 1. Consider the graphs of the functions

$$u_c(t),\ 1 - u_c(t),\ u_c - u_{2c}.$$

These are as in the next figures.

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We can easily compute the Laplace Transform of $u_c(t)$ as follows

$$\mathcal{L}(u_c(t)) \ = \ \int_0^\infty e^{-st} u_c(t) dt$$

$$
\begin{aligned}
&= \int_c^\infty e^{-st}dt \\
&= -\frac{1}{s}e^{-st}\Big|_{t=c}^{\infty} \\
&= \frac{e^{-cs}}{s} \text{ for } s > 0
\end{aligned}
$$

For a function $f(t)$ defined for $t > 0$, consider the function $g(t) = u_c(t)f(t-c)$. The graph of $g(t)$ is zero for $0 < t < c$, and the graph of $f(t)$ translated to the right to start at c.

The Laplace Transform of $g(t)$ is simply related to that of f.

Theorem. *If $F(s) = \mathcal{L}(f(t))$ exists for $s > a \geq 0$, and c is a positive constant, then*

$$\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)) = e^{-cs}F(s)$$

for $s > a$. Also,

$$\mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(t)f(t-c).$$

Proof.

Let $F(s) = \mathcal{L}(f(t))$.

We have, using definitions and the the substitution $\xi = t - c$,

$$\begin{aligned}
\mathcal{L}(u_c(t)f(t-c)) &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\
&= \int_c^\infty e^{-st}f(t-c)dt \\
&= \int_0^\infty e^{-s(c+\xi)}f(\xi)d\xi \\
&= e^{-sc}\int_0^\infty e^{-s\xi}f(\xi)d\xi \\
&= e^{-sc}F(s).
\end{aligned}$$

QED.

An alternative formulation of the above theorem is the following.

$$\mathcal{L}(u_c(t)f(t)) = e^{-cs}G(s)$$

where $G(s) = \mathcal{L}(f(t+c))$.

To see this, just write $f(t) = f(t+c-c)$ and use the above theorem.

Let us present some examples of the use of these formulas.

Example 2. Let $f(t)$ be defined by

$$f(t)\begin{cases} sin(t) & \text{if } 0 < t < \frac{\pi}{4} \\ sin(t) + cos(t - \frac{\pi}{4}) & \text{if } t \geq \frac{\pi}{4} \end{cases}$$

Compute $\mathcal{L}(f(t))$.

We have $f(t) = sin(t) + u_{\frac{\pi}{4}}cos(t - \frac{\pi}{4})$.

So,

$$\begin{aligned}\mathcal{L}(f(t)) &= \mathcal{L}(sin(t)) + \mathcal{L}(u_{\frac{\pi}{4}} cos(t - \frac{\pi}{4})) \\ &= \frac{1}{s^2+1} + \frac{se^{-s\frac{\pi}{4}}}{s^2+1}.\end{aligned}$$

Example 3. Find the inverse transform of

$$F(s) = \frac{1-e^{-2s}}{s^2}.$$

We have

$$\begin{aligned}\mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}(\frac{1}{s^2}) - \mathcal{L}^{-1}(\frac{e^{-2s}}{s^2}) \\ &= t - u_2(t)(t-2).\end{aligned}$$

Example 4. Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

We have

$$\begin{aligned}G(s) &= \frac{1}{s^2-4s+5} \\ &= \frac{1}{s^2-4s+4+1} \\ &= \frac{1}{(s-2)^2+1} \\ &= F(s-2)\end{aligned}$$

where

$$F(s) = \frac{1}{s^2+1}.$$

So,

$$F(s) = \mathcal{L}(sin(t))$$

and

$$G(s) = \mathcal{L}(e^{2t}sin(t)).$$

Example 5. Find $\mathcal{L}^{-1}(\frac{2se^{-3s}}{s^2+5})$.
We first find $\mathcal{L}^{-1}(\frac{2s}{s^2+5})$.

$$\begin{aligned} \mathcal{L}^{-1}(\frac{2s}{s^2+5}) &= \mathcal{L}^{-1}(\frac{2s}{s^2+(\sqrt{5})^2}) \\ &= 2\cos(\sqrt{5}t) \end{aligned}$$

Then, we get

$$\mathcal{L}^{-1}(\frac{2se^{-3s}}{s^2+5}) = 2u_3(t)\cos(\sqrt{5}(t-3)).$$

Example 6. Find $\mathcal{L}(u_2(t)t^2)$.
We use the formula

$$\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c)).$$

We get

$$\begin{aligned}\mathcal{L}(u_2(t)t^2) &= e^{-2s}\mathcal{L}((t+2)^2)\\ &= e^{-2s}(\mathcal{L}(t^2)+4\mathcal{L}(t)+4\mathcal{L}(1))\\ &= e^{-2s}(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s})\end{aligned}$$

Differential Equations with discontinuous right hand sides

We apply the above techniques to some initial value problems.

Example 6

Solve the initial value problem

$$2y''+y'+2y = u_5(t)-u_{20}(t), y(0)=0,\ y'(0)=0.$$

From our formulas we have

$$\begin{aligned}\mathcal{L}(y) &= \frac{\mathcal{L}(u_5(t))_{-}\mathcal{L}(u_{20}(t))}{2s^2+s+2}\\ &= \frac{e^{-5s}-e^{-20s}}{s(2s^2+s+2)}\end{aligned}$$

and we have to take the inverse Laplace transform of the right side.

This is

$$\mathcal{L}^{-1}(\frac{e^{-5s}}{s(2s^2+s+2)})-\mathcal{L}^{-1}(\frac{e^{-20s}}{s(2s^2+s+2)}).$$

This is

$$u_5(t)f(t-5) - u_{20}(t)f(t-20)$$

where

$$f(t) = \mathcal{L}^{-1}(\frac{1}{s(2s^2+s+2)}),$$

We now compute this last inverse transform.
Using partial fractions, we write

$$\frac{1}{s(2s^2+s+2)}) = \frac{A}{s} + \frac{B+Cs}{2s^2+s+2}$$

We determine A, B, C from

$$2As^2 + As + 2A + Bs + Cs^2 = 1 \ \ \forall\ s.$$

Setting terms of like powers equal we get a system of three equations in the unknowns A, B, C, which we solve to get

$$A = \frac{1}{2},\ B = -\frac{1}{2},\ C = -1.$$

Hence, we have

$$(\frac{1}{2})\frac{1}{s} - \frac{\frac{1}{2}+s}{2(s^2+\frac{s}{2}+1)} = (\frac{1}{2})\frac{1}{s} - \frac{\frac{1}{4}+s+\frac{1}{4}}{2((s+\frac{1}{4})^2+\frac{15}{16})}$$

We compute the inverse Laplace transform of this as the sum of the terms

$$\mathcal{L}^{-1}(\frac{1}{2s}) - \mathcal{L}^{-1}(\frac{\frac{1}{4}}{2((s+\frac{1}{4})^2 + \frac{15}{16})}) - \mathcal{L}^{-1}(\frac{s+\frac{1}{4}}{2((s+\frac{1}{4})^2 + \frac{15}{16})})$$

We have

$$\mathcal{L}^{-1}(\frac{1}{2s}) = \frac{1}{2},$$

$$\begin{aligned}
\mathcal{L}^{-1}(\frac{\frac{1}{4}}{2((s+\frac{1}{4})^2 + \frac{15}{16})}) &= \mathcal{L}^{-1}(\frac{1}{8\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{((s+\frac{1}{4})^2 + \frac{15}{16})} \\
&= \frac{1}{2\sqrt{15}}e^{-\frac{t}{4}}\sin(\sqrt{\frac{15}{16}}t),
\end{aligned}$$

and

$$\mathcal{L}^{-1}(\frac{s+\frac{1}{4}}{2((s+\frac{1}{4})^2 + \frac{15}{16})}) = \frac{1}{2}e^{-\frac{t}{4}}\cos(\sqrt{\frac{15}{16}}t).$$

Impulse functions

In some cases one wants to consider a function which is very large for a short amount of time. One wants to take integrals of these functions and to consider them as forcing terms in differential equations.

The standard impulse function is the Dirac delta function. This is thought of as a function $\delta(t)$ which is infinite at $t = 0$, zero at $t \neq 0$, and has integral with value 1. There is no classical function which these properties, so it takes some work to make rigorous sense of this. The mathematical theory of distributions (invented by the French mathematician Laurent Schwartz) is the modern way to rigorously justify the concepts and calculations involving Dirac delta functions. This theory is studied in advanced analysis courses and is beyond the scope of this course.

Nevertheless, one can operate formally with delta functions by assuming that they have certain properties. So, we assume that the expression $\delta(t-t_0)$ represents a "generalized function" whose value is 0 for $t \neq t_0$, infinite at $t = t_0$, and

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0)$$

for any continuous function $f(t)$.

We assume that the Laplace transform $\mathcal{L}(\delta(t-t_0))$ is defined by the formula

$$\mathcal{L}(\delta(t-t_0)) = e^{-st_0}.$$

Using this one can solve differential initial value problems of the form

$$ay'' + by' + cy = A\delta(t-t_0), \; y(0) = 0, \; y'(0) = 0$$

as we did above with Laplace transform methods. (Here A is a real constant).

One gets

$$\mathcal{L}(y) = A\frac{\mathcal{L}(\delta(t-t_0))}{as^2+bs+c}.$$

Using $\mathcal{L}(\delta(t-t_0)) = e^{-st_0}$, we can find the inverse Laplace transform and find y in terms of Heaviside functions as above.

Convolutions.

It is sometimes desirable to compute the inverse Laplace transform of the product of two functions $F(s)$ and $G(s)$.

This calculation requires an operation on functions called *convolution.*

Given $f(t),\ g(t)$ two piecewise continuous functions of exponential order a defined on $(0,\infty)$, we define

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

The function $f \star g$ is called the convolution of f and g. It will also have exponential order a.

There are the following properties.

1. $f \star g = g \star f$ (commutativity of $\star$)
2. $f \star (g \star h) = (f \star g) \star h$ (associativity of $\star$)
3. $f \star (g_1 + g_2) = f \star g_1 + f \star g_2$ (linearity of $\star$)
4. $f \star 0 = 0$

5. If $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$, then

$$\mathcal{L}((f \star g)(t)) = F(s)G(s)$$

(product rule for $\star$)

The first four properties follow directly from well-known properties of integrals.

The product rule is an application of change of order of integration in an improper double integral.

Let us describe this.

From the definition of $\mathcal{L}(f \star g)$ we have

$$\begin{aligned}
\mathcal{L}(f \star g) &= \int_0^\infty e^{-st}(f \star g)(t)dt \\
&= \int_0^\infty e^{-st}(\int_0^t f(t-u)g(u)du)dt \\
&= \int_0^\infty (\int_0^t e^{-st} f(t-u)g(u)du)dt
\end{aligned}$$

Let $h(t,u) = e^{-st}f(t-u)g(u)$.

We think of the last integral as the interated integral of a double integral in which we first fix $t = t_0$, then integrate respect to the variable u as u runs from u=0 to $u = t_0$ along the vertical line through $(t_0, 0)$ and finally let t_0 run from 0 to ∞. This is a double integral of the function $h(t,u)$ over the region in the (t,u) plane bounded by the lines $u = 0$ and $u = t$. We can reverse the order of integration, fixing $u = u_0$, integrating with respect to t as t runs from u_0 to ∞ along the line $u = u_0$, and finally letting the horizontal lines $u = u_0$ go from $u_0 = 0$ to $u_0 = \infty$.

Thus, we can rewrite the last integral as

$$\int_{u=0}^{u=\infty} (\int_{t=u}^{t=\infty} e^{-st} f(t-u) g(u) dt) du$$

Let us change variables in the last inside integral, setting $v = t - u$, thinking of u as a constant.

Then, $dv = dt$, and the iterated integral becomes

$$\int_{u=0}^{u=\infty} (\int_{v=0}^{v=\infty} e^{-s(u+v)} f(v) g(u) dv) du$$

which is equal to

$$\int_{u=0}^{u=\infty} (\int_{v=0}^{v=\infty} e^{-su} e^{-sv} f(v) g(u) dv) du.$$

The terms in this integral involving v separate from those involving u, and the integral becomes the product

$$(\int_{v=0}^{v=\infty} e^{-sv} f(v) dv)(\int_{u=0}^{u=\infty} e^{-su} g(u) du)$$

This is just $\mathcal{L}(f(t))\mathcal{L}(g(t))$.

Some more examples of Laplace transforms:

Find the Laplace transforms of the following functions.

1.

$$f(t) = \begin{cases} 0 & t < 3 \\ (t-3)^2 & t \geq 3 \end{cases}$$

2.

$$f(t) \;=\; \begin{cases} 0 & t < 3 \\ t^2 - 6t + 12 & t \geq 3 \end{cases}$$

Solution for the first function.
We have

$$f(t) = u_3(t)(t-3)^2.$$

Using the formula for $u_c(t)f(t-c)$, we get

$$e^{-3s}F(s)$$

where

$$F(s) = \mathcal{L}(t^2) = \frac{2}{s^3}$$

Solution for the second function.
We have

$$f(t) = u_3(t)((t-3)^2+3) = u_3(t)((t-3)^2) + u_3(t)3.$$

We get the Laplace transform is

$$e^{-3s}\frac{2}{s^3} + 3\frac{e^{-3s}}{s}.$$

Impulse response functions

Convolutions give us a way of representing quite general inverse Laplace transforms as certain integrals. They are mainly of theoretical use because it may be very difficult to actually compute the integrals involved.

Consider the initial value problem (IVP) for the general second order non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), y(0) = y_0, y'(0) = y_0' \quad (1)$$

If we find the solution $y_h(t)$ of the *homogeneous* IVP

$$y'' + p(t)y' + q(t)y = 0, y(0) = y_0, y'(0) = y_0' \quad (2)$$

and a particular solution $y_1(t)$ of the IVP (with initial conditions $y(0) = 0, y'(0) = 0$)

$$y'' + p(t)y' + q(t)y = g(t), y(0) = 0, y'(0) = 0, \quad (3)$$

then the sum $y_h(t) + y_1(t)$ solves the original IVP.

In the case of constant coefficients (i.e., $p(t) = b$, $q(t) = c$ with b and c constants), the method of *Variation of Parameters* applies to allow us to write the solution to (1) in terms of an integral involving $g(t)$ and the first and second fundamential solutions of $y'' + py' + qy = 0$.

For functions $g(t)$ which have Laplace transforms, there is an alternate way of solving the IVP (1) using convolutions.

To discuss this method, we make the following definition.

Consider a second degree polynomial $z(s) = as^2+bs+c$ with a, b, c constants and $a \neq 0$.

The *impulse-reponse function* of $z(s)$ is the inverse Laplace transform $\mathcal{L}^{-1}(\frac{1}{z(s)})$.

This is a very special function of t which depends on the roots of $z(s)$.

To see what these functions can be involves the various cases for the roots.

Case 1: (real distinct roots) $z(s) = a(s-r_1)(s-r_2)$, $r_1 \neq r_2$, both real.

Using partial fractions and writing

$$\frac{1}{a(s-r_1)(s-r_2)} = \frac{A}{s-r_1} + \frac{B}{s-r_2}$$

we get that

$$\mathcal{L}^{-1}\left(\frac{1}{z(s)}\right) = Ae^{r_1 t} + Be^{r_2 t}$$

Case 2: (real multiple root) $z(s) = a(s-r_1)^2$

We have

$$\mathcal{L}^{-1}\left(\frac{1}{z(s)}\right) = \frac{te^{r_1 t}}{a}$$

Case 3: (complex roots) $z(s) = a(s-(\alpha+i\beta)(s-(\alpha-i\beta))$ with $\beta \neq 0$

We have

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{z(s)}\right) &= \frac{1}{a}\frac{1}{(s-\alpha-i\beta)(s-\alpha+i\beta)} \\
&= \frac{e^{\alpha t}}{a}\mathcal{L}^{-1}\left(\frac{1}{(s-i\beta)(s+i\beta)}\right) \\
&= \frac{e^{\alpha t}}{a}\mathcal{L}^{-1}\left(\frac{1}{s^2+\beta^2}\right) \\
&= \frac{e^{\alpha t}}{a\beta} sin(\beta t)
\end{aligned}$$

The following *Solution Decomposition Theorem* gives us a theoretical way to write solutions of non-homogeneous second order equations with constant coefficients using Laplace transform techniques and convolutions.

Theorem (Solution Decomposition Theorem) *Consider the IVP*

$$ay'' + by' + cy = g(t),\ \ y(0) = y_0,\ \ y'(0) = y_0' \qquad (4)$$

where a, b, c, y_0, y_0' *are constants with* $a \neq 0$ *and* $g(t)$ *is a function defined for* $t > 0$ *which has a Laplace transform.*

Let $z(s) = as^2 + bs + c$ *be the characteristic polynomial of (4) (as a function of* s*) and let*

$$y_\delta = y_\delta(t) = \mathcal{L}^{-1}\left(\frac{1}{z(s)}\right)$$

be the impulse respsonse function of $z(s)$.

Then the unique solution to (4) can be written as

$$y(t) = y_h(t) + y_\delta * g$$

where $y_h(t)$ is the unique solution of the homogenous IVP

$$ay'' + by' + cy = 0,\ \ y(0) = y_0,\ \ y'(0) = y'_0$$

Proof. The proof of this theorem is easy.

We have that the solution $y(t)$ of (4) satisfies

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{(as+b)y_0 + ay'_0 + \mathcal{L}(g(t))}{z(s)}\right] \\ &= \mathcal{L}^{-1}\frac{(as+b)y_0 + ay'_0}{z(s)} + \mathcal{L}^{-1}\left(\frac{\mathcal{L}(g(t))}{z(s)}\right) \end{aligned}$$

The first term $y_h(t) = \mathcal{L}^{-1}\left[\frac{(as+b)y_0+ay'_0}{z(s)}\right]$ clearly satisfies the homogeneous IVP, and the second term is the inverse transform of the product of $\frac{1}{z(s)}$ and $\mathcal{L}(g(t))$.

By the convolution theorem this last term equals $y_\delta * g$. QED.