12c. The Euler Equation

The special second order equation

\[ ax^2 y'' + xby' + cy = g(x) \]  

where \( a \) and \( b \) are constants, \( a \neq 0 \), and \( g(x) \) is continuous is called an Euler differential equation. It is linear of the second order.

Once we have found two linearly independent solutions, \( y_1(x), y_2(x) \) of the associated homogeneous equation, we can find the general solution of (1) by the method of Variation of Parameters.

Dividing by \( a \), we can write the equation as

\[ x^2 y'' + xpy' + qy = 0 \]  

where \( p = \frac{b}{a}, \ q = \frac{c}{a} \).

We proceed to find two independent solutions.

Let us begin by trying to find a solution of the form \( y(x) = x^r \) for some real number \( r \) and non-zero \( x \). (It will turn out later, that we actually have to consider complex \( r \) in some cases). Let us note that, for a real or complex number \( r \) and a positive \( x \), the complex power \( x^r \) is defined by

\[ x^r = \exp(r \ln(x)). \]

Returning to the Euler equation, we have

\[ x^2 r(r - 1)x^{r-2} + x p \, r x^{r-1} + q \, x^r = r(r - 1) \, x^r + p \, r x^r + q \, x^r = 0 \]

for all \( x \), so

\[ F(r) = r(r - 1) + pr + q = 0 \]  

That is, \( r \) must be a root of the poly \( F(r) = r(r - 1) + pr + q = r^2 + (p - 1)r + q \). The equation (4) is called the indicial equation (for the Euler equation).

We have three cases:

Case 1: \( F(r) = (r - r_1)(r - r_2) \) where \( r_1, r_2 \) are real and distinct. In this case, one can verify by reversing the above steps that both \( y_1(x) = x^{r_1} \) and \( y_2(x) = x^{r_2} \) are solutions to (2).
The general solution is then,

\[ y(x) = c_1 x^{r_1} + c_2 x^{r_2}, \quad x > 0 \]

Case 2: \( F(r) = (r - r_1)^2 \) where \( r_1 \) is real.

As above, we get one non-zero solution \( y_1(x) = x^{r_1} \).

We claim:

A second linearly independent solution is

\[ y_2(x) = x^{r_1} \ln(x). \quad (5) \]

Once this is done, the general solution has the form

\[ c_1 x^{r_1} + c_2 x^{r_1} \ln(x), \quad x > 0 \]

We justify (5) by the method of reduction of order from Section 8.

First, observe that \( F(r) = (r - r_1)^2 \), so \( F'(r_1) = 0 \). That is, \( r_1 \) is a root of \( F'(r) \).

Now,

\[ F(r) = r^2 + (p - 1)r + q \]

and

\[ F'(r) = 2r + (p - 1). \]

Since \( F'(r_1) = 0 \), we get

\[ 2r_1 + p = 1 \quad (6) \]

Now, recall the method of reduction of order from from Section 8 equations (1) and (4).

Given a non-zero solution \( y_1(x) \) to the homogeneous linear second order differential equation
\[ y'' + P(x)y' + Q(x)y = 0, \]

we can find a second, linearly independent, solution \( y(x) = y_1(x)v(x) \) where \( v(x) \) is a non-constant function and satisfies

\[
v(x) = \int e^{\exp \left( \int \left[ -\frac{2y_1'}{y_1} - P \right] dx \right)} dx
\]

Applying this to the Euler equation, with \( p, q \) constants, we have

\[
x^2y'' + xy' + qy = 0
\]

\[
y'' + \frac{p}{x}y' + \frac{q}{x^2}y = 0,
\]

\[ y_1(x) = x^{r_1} \]

\[ P(x) = \frac{p}{x}, \]

\[
\frac{2y_1'}{y_1} = -\frac{2r_1 x^{r_1-1}}{x^{r_1}} = -\frac{2r_1}{x}
\]

\[ -P = -\frac{p}{x}, \]

so,

\[
v(x) = \int e^{\exp \left( \int \left[ -\frac{2r_1}{x} - \frac{p}{x} \right] dx \right)} dx
\]

Using (6) (i.e., \( 2r_1 + p = 1 \)) we have
\[ v(x) = \int \exp \left( \int \left[ \frac{-1}{x} \right] dx \right) dx \]

or

\[ v(x) = \int \exp(\ln(1/x))dx = \int \frac{1}{x}dx = \ln(x) \]

We don’t need the constants of integration.

Thus, our second independent solution is \( y_2(x) = x^{r_1} \ln(x) \) and (5) is justified.

Case 3: the roots of \( F(r) \) are complex conjugates:

\[ r_1 = a + bi, \quad r_2 = a - bi. \]

In this case, we get a complex solution of the form

\[ y_c = x^{ri} = x^{a+bi}, \quad x > 0 \]

The real and imaginary parts give two linearly independent real solutions.

Thus, denoting the real and imaginary parts of \( z \) by \( Re(z), Im(z) \), we get

\[ y_1(x) = Re(x^{a+b i}) \]
\[ = x^a \cdot Re(x^{b i}) \]
\[ = x^a \cdot Re(exp(b \cdot i \ln(x))) \]
\[ = x^a \cdot \cos(b \cdot \ln(x)) \]

and
\[ y_2(x) = \text{Im}(x^{a+b} i) = x^a \text{Im}(x^b i) = x^a \text{Im}(\exp(b i \ln(x))) = x^a \sin(b \ln(x)) \]

Hence, the general solution is

\[ y(x) = x^a(c_1 \cos(b \ln(x)) + c_2 \sin(b \ln(x)) \], \ x > 0

Remark. The indicial equation for the Euler equation with \( a \neq 1 \).

Notice that, with \( p = \frac{b}{a}, \ q = \frac{c}{a} \), we have

\[ r(r - 1) + pr + q = r^2 - r + pr + q = r^2 - r + \frac{b}{a}r + \frac{c}{a} \]

and multiplying through by \( a \), we get

\[ G(r) = ar^2 - ar + br + c = ar^2 + (b - a)r + c \quad (7) \]

which has the same roots as \( F(r) \). Using (7) saves one algebraic step in finding the roots of the indicial equation when \( a \neq 1 \).

Example 1

Find the general solution to the equation

\[ x^2 y'' + 6x y' + 6y = 0 \]

The indicial equation is

\[ F(r) = r(r - 1) + 6r + 6 = r^2 + 5r + 6 = (r + 3)(r + 2) \]

Hence, the general solution is

\[ y(x) = \frac{C_1}{x^3} + \frac{C_2}{x^2}, \ x > 0 \]

Example 2
Solve the initial value problem

\[ x^2 y'' + 6xy' + 6y = 0, \quad y(1) = 2, \quad y'(1) = 3 \]

We saw that the general solution is

\[ y(x) = C_1 x^{-3} + C_2 x^{-2} \]

So,

\[ y(1) = 2 = C_1 + C_2 \]

and

\[ y'(1) = 3 = -3C_1 - 2C_2 \]

So, \( c_1 = -7, C_2 = 9 \), and the answer is

\[ y(x) = -7x^{-3} + 9x^{-2} \]

**Example 3**

Find the general solution to the equation

\[ x^2 y'' + 8xy' + 9y = 0 \]

The indicial equation is \( F(r) = r^2 + 7r + 9 \).

The roots are

\[ r = \frac{1}{2}(-7 \pm \sqrt{49 - 36}) = -\frac{7}{2} \pm \left(\frac{\sqrt{13}}{2}\right) \]

General solution:

\[ y(x) = C_1 x^{r_1} + C_2 x^{r_2} \]

where \( r_1 = -\frac{7}{2} - (\frac{\sqrt{13}}{2}) \), \( r_2 = -\frac{7}{2} + (\frac{\sqrt{13}}{2}) \)

**Example 4**

Find the general solution to the equation

\[ x^2 y'' + 2xy' + 10y = 0 \]

Inicial equation: \( r^2 + r + 10 \).

Roots: \( r = \frac{1}{2}(-1 \pm \sqrt{1 - 40}) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{39}i \).

General solution:
$$y(x) = x^{-\frac{1}{2}} \left( C_1 \cos \left( \frac{1}{2} \sqrt{39} \ln(x) \right) + C_2 \sin \left( \frac{1}{2} \sqrt{39} \ln(x) \right) \right)$$

**Example 5**

Find the general solution to the equation

$$2x^2 y'' + 3xy' - 4y = 0$$

Initial polynomial: $2r^2 + r - 4$.

Roots: $r_{\pm} = \frac{1}{2} (-1 \pm \sqrt{33})$.

General solution:

$$y(x) = C_1 x^{r_+} + C_2 x^{r_-}$$