Variation of Coasts $L(y) = g$

$y'' + p(t)y' + q(t)y = g(t)$

Find $y_1, y_2$ solving $L(y) = 0$

First $y_p = y_1 v_1 + y_2 v_2$

\[
\begin{pmatrix}
y_1'' \\
y_2''
\end{pmatrix}
\begin{pmatrix}
v_1' \\
v_2'
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
g(t)
\end{pmatrix}
\]

$v_1' = \frac{y_2 q}{W}$, $v_2' = \frac{y_1 q}{W}$

$v_1' = -\frac{y_2 q}{W}$, $v_2' = \frac{y_1 q}{W}$
\[ y'' + 25y = 30 \sec (5t) \]
\[ y(0) = y_0, \quad y'(0) = y_0' \]

\[ r^2 + 25 = 0 \]
\[ r = \pm 5i, \quad i = \sqrt{-1} \]

\[ y_1 = \cos (5t), \quad y_2 = \sin (5t) \]

Try \( y = y_p = v_1 \cos (5t) + v_2 \sin (5t) \)

\[
\begin{pmatrix}
(y_1, y_2)' / v_1'
\end{pmatrix} = 
\begin{pmatrix}
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
(y_1', y_2') / v_2'
\end{pmatrix} = 
\begin{pmatrix}
-30 \sec (5t)
\end{pmatrix}
\]

\[
\begin{cases}
v_1' = 30 \sin (5t) \sec (5t) / 5 \\
v_2' = (\cos (5t) (-30) \sec (5t)) / 5
\end{cases}
\]

\[ W = \begin{vmatrix}
\cos (5t) & \sin (5t) \\
-5 \sin (5t) & 5 \cos (5t)
\end{vmatrix} = 5 \]
10. Particular Solutions of Non-homogeneous second order equations—Variation of Parameters

As above, we consider the second order differential equation

\[ L(y) = y'' + p(t)y' + q(t)y = g(t) \]  \hspace{1cm} (1)

where \( p, q, g \) are continuous functions in an interval \( I \).

In the method called variation of parameters, we start with two linearly independent solutions \( y_1, y_2 \) of \( L(y) = 0 \), and we look for a particular solution \( y(t) \) of \( L(y) = g \) of the form

\[ y(t) = y_1(t)v_1(t) + y_2(t)v_2(t) \]  \hspace{1cm} (2)

where \( v_1 \) and \( v_2 \) are not constant functions.

It turns out that we can consider the system of linear equations

\[ y_1v_1' + y_2v_2' = 0 \]  \hspace{1cm} (3)
\[ y_1'v_1 + y_2'v_2 = g(t) \]  \hspace{1cm} (4)

We solve these equations for \( v_1' \) and \( v_2' \) and integrate to get \( v_1 \) and \( v_2 \).
\[ v_1' = \frac{30 \sin(5t) \sec(5t)}{5} \]

\[ v_1 = \frac{1}{5} \int \frac{30 \sin(5t) \, dt}{\cos(5t)} \]

\[ u = \cos(5t), \quad du = -5 \sin(5t) \, dt \]

\[ = \frac{1}{5} \int 30 \left( -\frac{1}{5} \right) du \]

\[ = -\frac{6}{5} \ln|1u| = -\frac{6}{5} \ln|\text{abs}(\cos(5t))| \]

\[ v_2 = -6t \]

Ans: \[ y_p = -\frac{6}{5} \cos(5t) \ln(\text{abs}(\cos(5t))) \]

\[ -6t \sin(5t) \]
\[ y'' + 8y' + 16y = \frac{-2e^{-4t}}{t^2 + 1} \]

Find particular solution

\[ \text{Use of Var of params} \]

\[ r^2 + 8r + 16 = (r + 4)^2 \]

\[ y_1 = e^{-4t}, \quad y_2 = te^{-4t} \]

Want \( y = y_1y_1' + y_2y_2' \)

\[ \sigma_1 = -\frac{y_2g}{W} \]

\[ \sigma_2 = \frac{y_1g}{W} \]

\[ W = y_1y_2' - y_2y_1' = e^{-4t}(e^{-4t} + 4te^{-4t}) - e^{-4t}(-4t - 4te^{-4t}) \]

\[ W = e^{-8t} \]
\[ U_1 = -\frac{A}{2} \ln (t^2 + 1) \]

\[ U_2 = A \arctan (t) \]

\[ y'' - 2y' + y = 384 e^{9t} \]

\[ (r-1)^2 \]

\[ y_1 = e^t, \quad y_2 = te^t \]

\[ y = Ae^{-18t} + \frac{384}{81-18t} \]

6 = \frac{384}{64}
Let us verify that if the equations (3) and (4) are satisfied, then (2) does indeed give us a particular solution. Let us begin by observing that differentiating the first equation above gives

\[(y_1v_1' + y_2v_2')' = y_1'v_1' + y_2'v_2' + y_1v_1'' + y_2v_2'' = 0\]  \hspace{1cm} (5)

Hence, using (3), (4), and (5), we get

\[y'' + py' + qy = (y_1v_1 + y_2v_2)'' + p(y_1v_1 + y_2v_2)' + q(y_1v_1 + y_2v_2)
\]
\[= y_1''v_1 + 2y_1'v_1' + y_1v_1'' + y_2''v_2 + 2y_2'v_2' + y_2v_2'' + p(y_1'v_1 + y_1v_1' + y_2'v_2 + y_2v_2') + q(y_1v_1 + y_2v_2)
\]
\[= v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'(2y_1' + py_1) + v_2'(2y_2' + py_2) + v_1''y_1 + v_2''y_2
\]
\[= y_1'v_1' + y_2'v_2' + p(y_1v_1' + y_2v_2') + y_1'v_1' + y_2'v_2' + v_1''y_1 + v_2''y_2
\]
\[= y_1'v_1' + y_2'v_2'\]
\[= g(t)\]

We wish to put the solution of the linear system (3), (4) in a simple form. For this purpose, it is useful to discuss a formula for the solution of linear equations known as
Cremer’s rule. We will consider this here in the special case of two linear equations in two unknowns.

We first consider simple properties of the determinant as a function of the columns of the matrix.

**Properties of Determinants of $2 \times 2$ matrices:**

Let

$$
\begin{vmatrix}
    a & b \\
    c & d
\end{vmatrix} = ad - bc
$$

denote the determinant of the matrix

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}.
$$

For any real numbers $a, b, c, d, e, f \alpha$ we have

$$
\begin{vmatrix}
    \alpha a & b \\
    \alpha c & d
\end{vmatrix} = \alpha \begin{vmatrix}
    a & b \\
    c & d
\end{vmatrix},
$$

$$
\begin{vmatrix}
    a + e & b \\
    c + f & d
\end{vmatrix} = \begin{vmatrix}
    a & b \\
    c & d
\end{vmatrix} + \begin{vmatrix}
    e & b \\
    f & d
\end{vmatrix},
$$

and

$$
\begin{vmatrix}
    a & b \\
    c & d
\end{vmatrix} = - \begin{vmatrix}
    b & a \\
    d & c
\end{vmatrix},
$$

Assume that $x, y$ satisfy the system
\[ ax + by = e \]
\[ cx + dy = f. \]

Then,
\[
\begin{vmatrix}
  e & b \\
  f & d
\end{vmatrix}
= \begin{vmatrix}
  ax + by & b \\
  cx + dy & d
\end{vmatrix}
= \begin{vmatrix}
  ax & b \\
  cx & d
\end{vmatrix}
+ \begin{vmatrix}
  by & b \\
  dy & d
\end{vmatrix}
= x \begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix}
+ y \begin{vmatrix}
  b & b \\
  d & d
\end{vmatrix}
= x \begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix}.
\]

Similarly,
\[
\begin{vmatrix}
  a & e \\
  c & f
\end{vmatrix}
= y \begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix}.
\]

Now, if
\[
\begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix} \neq 0,
\]
then we can solve for \( x \) and \( y \) from the above equations to get
$x = \begin{vmatrix} e & b \\ f & d \\ a & b \\ c & d \end{vmatrix}, \quad y = \begin{vmatrix} a & e \\ c & f \\ a & b \\ c & d \end{vmatrix}$

Applying this to the system (3), (4), we get

$$v'_1 = \begin{vmatrix} 0 & y_2 \\ g & y'_2 \end{vmatrix} = \frac{-y_2 g}{W(y_1, y_2)}$$

$$v_1 = \int \frac{-y_2 g}{W(y_1, y_2)} \, dt$$

$$v'_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & g \end{vmatrix} = \frac{y_1 g}{W(y_2, y_2)}$$

$$v_2 = \int \frac{y_1 g}{W(y_1, y_2)} \, dt.$$  

**Example.**

Find the general solution to

$$y'' + 4y = 3 \csc(t).$$

The general solution to the the homogeneous equation is

$$y = A \cos(2t) + B \sin(2t).$$
We assume a particular solution of the form
\[ y = \cos(2t)v_1 + \sin(2t)v_2, \]
and we get the system
\[
\begin{align*}
\cos(2t)v'_1 + \sin(2t)v'_2 &= 0 \\
-2\sin(2t)v'_1 + 2\cos(2t)v'_2 &= 3\csc(t),
\end{align*}
\]
so,
\[ W(\cos(2t), \sin(2t)) = 2, \]
and
\[
\begin{align*}
v'_1 &= -3\frac{\sin(2t)\csc(t)}{2} \\
v'_2 &= 3\frac{\cos(2t)\csc(t)}{2}
\end{align*}
\]
This gives
\[
\begin{align*}
v_1 &= -\frac{3}{2} \int \sin(2t)\csc(t)dt \\
&= -\frac{3}{2} \int \frac{\sin(2t)}{\sin(t)}dt \\
&= -\frac{3}{2} \int \frac{2\sin(t)\cos(t)}{\sin(t)}dt \\
&= -3 \sin(t)
\end{align*}
\]
\[ v_2 = \frac{3}{2} \int \cos(2t) \csc(t) \, dt \]
\[ = \frac{3}{2} \int \frac{\cos(2t)}{\sin(t)} \, dt \]
\[ = \frac{3}{2} \int \frac{\cos(t)^2 - \sin(t)^2}{\sin(t)} \, dt \]
\[ = \frac{3}{2} \int \frac{1 - 2\sin(t)^2}{\sin(t)} \, dt \]
\[ = \frac{3}{2} \left( \log(\csc(t) - \cot(t)) + 3\cos(t) \right) \]

The general solution is

\[ y = A\cos(2t) + B\sin(t) + \cos(2t)(-3\sin(t)) \]
\[ + \sin(2t)\left( \frac{3}{2} \log(\csc(t) - \cot(t)) + 2\cos(t) \right). \]

Here is a second example.

\[ y'' - 3y' + 2y = 3te^{2t}. \]

The characteristic equation is \( r^2 - 3r + 2 = (r-2)(r-1) \) has roots \( r = 1, 2 \), so according to the table in section 9, in the method of undetermined coefficients, we need to assume the particular solution \( y_p \) has the form

\[ y_p = t(A + Bt)e^{2t}. \]
This involves a lot of computation to differentiate and plug into the d.e. to get the appropriate constants $A, B$.

Let us see what the method of Variation of Parameters gives us.

We get a solution $y_p = e^{2t}v_1 + e^tv_2$ where

\[
e^{2t}v'_1 + e^tv'_2 = 0
\]  
\[
2e^{2t}v'_1 + e^tv'_2 = 3te^{2t}
\]

This gives

\[
v'_2 = -e^tv'_1
\]

\[
2e^{2t}v'_1 + e^t(-e^tv'_1) = 3te^{2t}
\]

\[
e^{2t}v'_1 = 3te^{2t}
\]

\[
v'_1 = 3t, \quad v_1 = \frac{3t^2}{2}
\]

\[
v'_2 = -3te^t, \quad v_2 = -3(te^t - e^t).
\]

So, we get

\[
y_p = e^{2t}\frac{3t^2}{2} + e^t(-3)((te^t - e^t))
\]

\[
= e^{2t}\frac{3t^2}{2} - 3te^{2t} + 3e^{2t}
\]
Since $3e^{2t}$ is a solution of the homogeneous equation, the function

$$y_3 = e^{2t} \frac{3t^2}{2} - 3te^{2t} = te^{2t}(\frac{3t}{2} - 3)$$

is also a particular solution. This has the form given by the method of undetermined coefficients, and was obtained with much less computation than using that method directly. In our case, we only want a particular solution anyway.

Thus, in the d.e.

$$y'' + py' + qy = P_ne^{\alpha t}$$

with $p, q$ constants, the method of undetermined coefficients seems to give an improvement of the method of variation of parameters only if $P_n$ is a constant.