1. Find the general solution to each of the following differential equations.

(a) \( y' = t^3(y^2 - 1) \)

6 total pts

2 pts for separation

\[
\frac{dy}{y^2 - 1} = t^3 \, dt
\]

4 pts for correct integrals plus constant

\[
\frac{1}{2} \log\left(\frac{y - 1}{y + 1}\right) = \frac{t^4}{4} + C
\]

(b) \( y' + 2t^2y = 3t^2 \)

6 pts total

2 pts for correct integrating factor \( \mu = exp\left(\frac{2}{3}t^3\right) \).

4 pts for correct answer with constant

\[
y = \frac{3}{2} + C \exp\left(-\frac{2}{3}t^3\right)
\]
(c) \( y' + \frac{y}{t} = ty^3 \)

1 pts for recognition as Bernoulli with substitution \( v = y^{-2} \)

1 pt for correct linear equation for \( v \): \( v' - (2/t)v = -2t \)

4 pts for solution to linear equation \( v = t^2(C - 2 * \log(t)) \)

2 pts for conversion back to \( y \): \( y = [t^2(C - 2 * \log(t))]^{-\frac{1}{2}} \)

(d) \( (3x^2y^2 + x)dx + (2x^3y + y^2)dy = 0. \)

2 pts for check \( M_y = N_x \) (or if just did correct method, add 2 pts to below)

2 pts for \( f(x, y) = x^3y^2 + x^2/2 + g(y) \)

2 pts for plugging in \( f_y = N \) to get \( g'(y) = y^2 \)

2 pts for ans: \( x^3y^2 + x^2/2y^3/3 = C \)
(e) $(9y^2 + 3 + 4xy^3)\,dx + (6xy + 3x^2y^2)\,dy = 0$  
10 pts total

4 pts for correct integrating factor: $\mu = x^2$

6 pts for finding correct solution: $x^4y^3 + 3x^3y^2 + x^3 = C$

2. The differential equation $y' = \frac{2xy}{x^2 + y^2}$ can be changed to a separable differential equation by an appropriate substitution. Write both this substitution and the resulting separable differential equation. (You do not have to solve the resulting differential equation).

4 pts total

2 pts for correct substitution: $y = xv$

2 pts for correct separable equation: $v + xv' = \frac{2v}{1 + v^2}$
3. A certain radioactive substance decays from 10g to 5g in 20 years.

(a) Write a differential equation for the amount of the radioactive substance at time $t$.

4 pts total

2 pts for correct equation: If $Q(t)$ is amount at time $t$, then $Q' = -kQ$

2 pts for finding $k$: $5 = 10e^{(-k20)}$

$\frac{1}{2} = e^{(-k20)}$, $-\log(2) = -k20$, $k = \frac{\log(2)}{20}$

(b) How long would it take for the substance to decay from 10g to 2g?

4 pts total

4 pts for $t_1$ with $Q(t_1) = 2$,

$2 = 10e^{(-kt_1)}$

$1/5 = e^{(-kt_1)}$

$-\log(5) = -kt_1$

$t_1 = \frac{\log(5)}{k} = \frac{20 \log(5)}{\log(2)}$ years
4. A 300 gallon tank contains 30 gallons of a 5 mg/gal salt solution. A solution with 2 mg/gal is introduced at the rate of 3 gal/min and the resulting solution is drained at the same rate.

(a) Write a differential equation for the amount $Q(t)$ of salt in the resulting solution at time $t$.

4 pts total

4 pts for DE: Let $Q(t)$ be the amount of salt in the tank at time $t$.

$$Q' = Q_{in} - Q_{out} = 6 - 3 \frac{Q}{30} = 6 - \frac{Q}{10}$$

$$Q' + \frac{Q}{10} = 6$$

(b) How much salt is in the combined mixture after 10 minutes?

6 pts total

4 pts for general solution of ODE:

$$Q = e^{-t/10} \left[ 60e^{t/10} + C \right] = 60 + Ce^{-t/10}$$

2 pts for determination of $C$: $Q_0 = 150 = 60 + C$, $C = 90$

2 pts for $Q(10)$: $Q(10) = (60 + \frac{90}{e})$ mg