Directions:

- Volunteers will be asked to present solutions in class.
- Each solution you present will count towards your final homework grade.

WARMUP PROBLEMS (Not to be turned in)

1. Compare and contrast the definitions of a limit, and continuity at a point. Specifically, what sort of domains do we consider when looking at continuity vs. limits?

HOMEWORK EXERCISES

1. [The sequential characterization of limits of functions] Suppose \( f : V \setminus \{a\} \to \mathbb{R}^m \), where \( V \) is an open subset of \( \mathbb{R}^n \), and \( a \in V \). Prove that \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{k \to \infty} f(x_k) = L \) for every sequence \( x_k \in V \setminus \{a\} \) that satisfies \( \lim_{k \to \infty} x_k = a \).

2. [The sequential characterization of continuity] Suppose \( f : E \to \mathbb{R}^m \). Prove that \( f \) is continuous at \( a \in E \) if and only if \( \lim_{k \to \infty} f(x_k) = f(a) \) for every sequence \( x_k \in E \) that satisfies \( \lim_{k \to \infty} x_k = a \). Compare with the results of problem 1.

3. Suppose \( f : E \to \mathbb{R}^m \), where \( E \subseteq \mathbb{R}^n \) is a closed set. Prove that \( f \) is continuous on \( E \) if and only if \( f^{-1}(C) \) is closed for every closed set \( C \subseteq \mathbb{R}^m \).

4. Consider \( f, g : \mathbb{R} \to \mathbb{R} \), where \( f(x) = \sin(x) \), and \( g(x) = x/|x| \) if \( x \neq 0 \), and \( g(0) = 0 \).
   
   (a) Define \( E_1 = (0, \pi) \), \( E_2 = [0, \pi] \), \( E_3 = (-1, 1) \) and \( E_4 = [-1, 1] \). For \( j = 1, \ldots, 4 \), compute \( f(E_j) \) and \( g(E_j) \). What conclusions can you draw about the images of connected/closed/open sets?
   
   (b) Define \( F_1 = (0, 1) \), \( F_2 = [0, 1] \), \( F_3 = (-1, 1) \) and \( F_4 = [-1, 1] \). For \( j = 1, \ldots, 4 \), compute \( f^{-1}(F_j) \) and \( g^{-1}(F_j) \). What conclusions can you draw about the inverse images of connected/closed/open sets?

5. Let \( H \) be a non-empty, compact subset of \( \mathbb{R}^n \).
   
   (a) If \( f : H \to \mathbb{R}^m \) is a function, we define
   
   \[
   \|f\|_H := \sup \{ \|f(x)\| : x \in H \}.
   \]

   Show that if \( f \) is continuous, then there exists an \( x^* \in H \) such that \( \|f\|_H = \|f(x^*)\| \).

   (b) Consider the following definition, which is the multi-variable extension of what you have already seen in the single variable case.
**Definition 1** We say a sequence of functions $f_k : H \to \mathbb{R}^m$ converge uniformly to $f : H \to \mathbb{R}^n$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for every $x \in H$ and $k \geq N$, we have $\|f_k(x) - f(x)\| < \epsilon$.

Show that $\|f_k - f\|_H \to 0$ if and only if $f_k$ converge to $f$ uniformly.

6. Show that $\|f_k - f\|_H \to 0$ if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that $k, j \geq N$ implies $\|f_k - f_j\|_H < \epsilon$.

7. Using the new topological tools that we have, construct a shorter proof of the following theorem, which appeared in homework 1: If $f \geq 0$ is continuous on $[a, b]$, prove that $\int_a^b f(x) \, dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

8. Suppose $H = [a, b] \times [c, d]$ is a rectangle, and $f : H \to \mathbb{R}$ is continuous, and $g : [a, b] \to \mathbb{R}$ is integrable. Prove that

$$F(y) = \int_a^b g(x)f(x, y) \, dx$$

is uniformly continuous on $[a, b]$. *Hint:* you may use the fact that $H$ is compact.