Directions:

- Volunteers will be asked to present solutions in class.
- Each solution you present will count towards your final homework grade.

HOMEWORK EXERCISES

1. Suppose $K$ is compact in $\mathbb{R}^n$, and $E \subseteq K$. Prove that $E$ is compact if and only if $E$ is closed.

2. Suppose $K$ is compact in $\mathbb{R}^n$, and for every $x \in K$, there is an $r = r(x) > 0$ such that $B_r(x) \cap K = \{x\}$. Prove that $K$ is a finite set.

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, and $K \subseteq \mathbb{R}^n$ is compact and connected. For each $x$, suppose there exists a $\delta = \delta(x) > 0$ such that $f(x) = f(y)$ for all $y \in B_{\delta(x)}(x)$. Prove that $f$ is constant on $K$.

   Note: This is an excellent example of a “local” to “global” result.

4. Recall that we defined the distance between a point $x \in \mathbb{R}^n$ and a set $A \subseteq \mathbb{R}^n$ as
   $$d(A, x) := \inf \{||x - y|| : y \in A\}$$

   Define the distance between two sets $A, B \subseteq \mathbb{R}^n$ as
   $$d(A, B) := \inf \{||x - y|| : x \in A, y \in B\}.$$  

   (a) Prove that if $A$ and $B$ are compact sets that satisfy $A \cap B = \emptyset$, then $d(A, B) > 0$.
   (b) Show that there exist nonempty, closed sets $A, B \subset \mathbb{R}^2$ such that $A \cap B = \emptyset$, but $d(A, B) = 0$.

5. Consider the function
   $$f(x, y) = \left(\frac{x - 1}{y - 1}, x + 2\right),$$
   whose domain is yet to be determined.

   (a) Identify the co-domain of $f$.
   (b) Find the largest domain $E$, where the above expression for $f$ makes sense. This is often considered the “natural domain”, or the “maximal domain” of $f$.  

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1In general, if you don’t assume that $K$ is connected, you can show that $f$ is constant on every connected component of $K$.

2I prefer the term “target space” in place of co-domain.
(c) Compute \( \lim_{(x,y) \to (1,-1)} f(x,y) \). You do not need a formal \( \epsilon - \delta \) proof if you cite the correct theorems.

6. Consider the function

\[
f(x, y) = \left( \frac{y \sin(x)}{x}, \tan\left(\frac{x}{y}\right), x^2 + y^2 - xy \right),
\]

whose domain is yet to be determined.

(a) Identify the co-domain of \( f \).

(b) As in the previous problem, find the natural domain of \( f \).

(c) Compute \( \lim_{(x,y) \to (1,-1)} f(x,y) \). Again, there is no need for a formal \( \epsilon - \delta \) proof.

7. Suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \), and \( \lim_{x \to a} f(x) = L \), where \( a \in \mathbb{R}^n \), and \( L \in \mathbb{R}^m \). Prove that there exists a constant \( M \) and an open set \( V \) with \( a \in V \) such that \( \|f(x)\| \leq M \) for all \( x \in V \).