Directions:

- Volunteers will be asked to present solutions in class.

**WARMUP PROBLEMS** (Not to be turned in)

1. Write down the definition of a partial derivative.

2. Write down the definition of the (total) derivative. Compare with your answer to part 1.

**HOMEWORK EXERCISES**

1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *independent of the second variable* if for each $x \in \mathbb{R}$, we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Show that $f$ is independent of the second variable if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)$. What is $\frac{\partial f}{\partial x}$ in terms of $g$?

2. If $f : \mathbb{R}^2 \to \mathbb{R}$, and $\frac{\partial f}{\partial x_2} = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$, show $f(x_1, x_2) = g(x_1)$ for some $g$. That is, show that $f$ is independent of the second variable. If $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$, show that $f$ is constant.

3. For each of the following functions $f$, find the matrix representation of a linear transformation $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m)$ such that

$$
\lim_{h \to 0} \frac{\|f(x + h) - f(x) - T(h)\|}{h} = 0.
$$

(a) $f(x) = (x^2, \sin(x))^T$.

(b) $f(x) = (e^x, x^{1/3}, 1 - x^2)^T$.

4. Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, and recall the definition of the *operator norm*,

$$
\|T\| := \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}.
$$

(a) Show that the supremum need only be taken over the unit sphere. That is, prove that $\sup_{\|x\|=1} \|T(x)\| = \|T\|$.

(b) Define

$$
m := \inf \left\{ C > 0 : \|T(x)\| \leq C\|x\| \quad \text{for all} \ x \right\}.
$$

Prove that $m = \|T\|$.
5. Suppose \( T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) is a linear function. Prove that \( DT(a) = T \) for all \( a \in \mathbb{R}^n \).

6. If \( f : \mathbb{R}^n \to \mathbb{R} \) is a scalar function, we define the directional derivative as
   \[
   D_v f(a) := \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t},
   \]
   provided this limit exists.
   
   (a) Show that \( D_{e_i} f(a) = \frac{\partial f}{\partial x_i}(a) \), provided both limits exist. Conclude that directional derivatives extend the definitions of partial derivatives.
   
   (b) If \( c \in \mathbb{R} \) is a scalar, show that \( D_c v f(a) = c D_v f(a) \), provided the directional derivative exists.
   
   (c) If \( f \) is differentiable at \( a \), show that \( D_v f(a) = Df(a)v \), where the left hand side denotes the directional derivative, and the right hand side denotes the multiplication of the total derivative against \( v \).

7. Two functions \( f, g : \mathbb{R} \to \mathbb{R} \) are considered to be equal up to nth-order at \( a \) if
   \[
   \lim_{h \to 0} \frac{f(a + h) - g(a + h)}{h^n} = 0.
   \]
   
   (a) Show that \( f \) is differentiable at \( a \) if and only if there is a function \( g \) of the form \( g(x) = a_0 + a_1(x - a) \) such that \( f \) and \( g \) are equal up to first order at \( x = a \).
   
   (b) If \( f'(a), f''(a), \ldots f^{(n)}(a) \) exist, show that \( f \) and the function \( g \) defined by
   \[
   g(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i
   \]
   are equal up to nth-order at \( a \). Hint: re-write the limit as \( \lim_{x \to a} \) and use L’Hôpital’s rule.

8. Suppose \( f : V \to \mathbb{R} \) is a function defined on an open set \( V \) containing the origin. Suppose further that \( |f(x)| \leq \|x\|^\alpha \), where \( \alpha > 1 \) is a scalar. Prove that \( f \) is differentiable at the origin. What happens when you assume \( \alpha = 1 \)?