This review sheet is a selection of problems that I think would provide good practice for you before the exam. I don’t guarantee that any of this will show up on the exam, nor do I guarantee that this is comprehensive. Your best bet for preparing for the exam is making sure you know how to do all of the homework assigned by the professor.

Integration

Calculate the following integrals. I also recommend that you do problems from the end of the integration chapter which are not categorized so that you can practice deciding which method to use.

Integration by Parts:

(i) \( \int x(\ln x)^2 \, dx \)

(ii) \( \int x^3 e^x \, dx \)

(iii) \( \int \arctan 2x \, dx \)

(iv) \( \int x \sin x \, dx \)

(v) \( \int x^n \sin x \, dx \)

(vi) \( \int e^{3x} \sin x \, dx \)

Partial Fraction Decomposition:

(i) \( \int \frac{1}{x(x+1)} \, dx \)

(ii) \( \int \frac{x}{(x-3)^2} \, dx \)

(iii) \( \int \frac{x^3}{(x-1)^2} \, dx \)

(iv) \( \int \frac{x^3}{x^4 + 4} \, dx \)

(v) \( \int \frac{x^2}{1 + x^2} \, dx \)

(vi) \( \int \frac{(x-2)^2 \arctan 2x - 12x^3 - 3x}{(4x^2 + 1)(x-2)^2} \, dx \)

Trigonometric Integrals:

(i) \( \int \sin^4 x \, dx \)

(ii) \( \int \cos^3 x \, dx \)

(iii) \( \int \sin^4 x \cos^3 x \, dx \)

(iv) \( \int \cos^2 x \, dx \)

(v) \( \int \frac{\sin^5 x}{\cos^3 x} \, dx \)

(vi) \( \int \frac{\sin 2x}{2 + \cos x} \, dx \)

Trigonometric Substitutions:

(i) \( \int \frac{dx}{x^2 \sqrt{x^2 + 1}} \)

(ii) \( \int \frac{x}{\sqrt{4 - x^2}} \, dx \)

(iii) \( \int \frac{dx}{(1 - x^2)^{\frac{3}{2}}} \)

(iv) \( \int \frac{dx}{16 + 4x^2} \)

(v) \( \int \frac{2}{\sqrt{t + 4t\sqrt{t}}} \, dx \)

(vi) \( \int \frac{x^2}{\sqrt{9 - x^2}} \, dx \)

General Cleverness:

(i) \( \int \sec x \, dx \)

(ii) \( \int \frac{1}{x^2 - 5x + 6} \, dx \)

(iii) \( \int \frac{\sqrt{x + 1}}{x} \, dx \)

(iv) \( \int \frac{\tan x}{(\ln(\cos x))^2} \, dx \)

(v) \( \int x \tan^2 2x \, dx \)

(vi) \( \int \frac{\sin(e^{-2x})}{e^2x} \, dx \)

Improper Integrals: Decide if the following integrals converge or diverge. If the integral can be evaluated, then do so, otherwise use the comparison test.

(i) \( \int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \, dx \)

(ii) \( \int_{0}^{\pi} \frac{\sin x}{1 + \cos x} \, dx \)

(iii) \( \int_{0}^{\infty} \frac{1}{\ln x} \, dx \)

(iv) \( \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \, dx \)

(v) \( \int_{0}^{\infty} x e^{-x} \, dx \)

(vi) \( \int_{1}^{\infty} x^7 \, dx \)
Series

Series for which we can find a sum. Decide if the following series converge and, if so, find the sum:

(i) \( \frac{3}{100} + \frac{3}{100^2} + \frac{3}{100^3} + \cdots \)
(ii) \( \sum_{n=0}^{\infty} e^n \)
(iii) \( \sum_{n=0}^{\infty} \sin n - \sin (n + 1) \)
(iv) \( 1 - 3 + 9 - 27 + \cdots \)
(v) \( \sum_{n=0}^{\infty} \frac{2^n}{6(5^n)} \)

Convergence. Decide whether the following series converge or diverge. State the test you use and make sure you check the hypotheses of each test.

(i) \( \sum_{n=1}^{\infty} \frac{7}{3^n} \)
(ii) \( \sum_{n=0}^{\infty} \frac{n}{n^2 + 3} \)
(iii) \( \sum_{n=1}^{\infty} \frac{\ln(n^2)}{n} \)
(iv) \( \sum_{n=1}^{\infty} e^{-2n} \)
(v) \( \sum_{n=1}^{\infty} \frac{1}{\ln(4)^n} \)
(vi) \( \sum_{n=0}^{\infty} \frac{n - 1}{n^3} \)
(vii) \( \sum_{n=1}^{\infty} \ln(n) \)
(viii) \( \sum_{n=1}^{\infty} \frac{3n^2}{n^2 + 3n + 1} \)

Taylor polynomials. Calculate the following Taylor polynomials:

(i) \( f(x) = \arctan x, \ n = 2, \ a = 0 \)
(ii) \( f(x) = \sqrt{1 + x}, \ n = 2, \ a = 0 \)
(iii) \( f(x) = \cos(2x + \frac{\pi}{2}), \ n = 2, \ a = \frac{\pi}{4} \)
(iv) \( f(x) = x^3 + x + 17, \ \text{do this for } n = 2, \ a = 1 \text{ and } a = 0 \).

Taylor series. Calculate the following Taylor series (the long way):

(i) \( f(x) = \sin(2x), \ a = 0 \)
(ii) \( f(x) = \cos \frac{x^3}{3}, \ a = 0 \)
(iii) \( f(x) = e^{-x}, \ a = 0 \)
(iv) \( f(x) = 5 \cos \pi x, \ a = 1 \)
(v) \( f(x) = \frac{1}{1-2x}, \ a = 0 \)
(vi) \( f(x) = \frac{1}{x}, \ a = 1 \)

Manipulation of Taylor series. Calculate the Taylor centered at 0 for the following functions using known Taylor series:

(i) \( f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \)
(ii) \( f(x) = \int \ln(1 + x) \, dx \)
(iii) \( f(x) = e^{-x} + 3x \)
(iv) \( f(x) = \frac{1}{1+x+x^2} \)
(v) \( f(x) = \frac{d}{dx} (xe^{-x}) \)
(vi) \( f(x) = \frac{3}{2 + 7x} \)

(v) has more than one method. Can you find both?

Using your knowledge of the geometric series, for which \( x \) will the Taylor series of (vi) converge?
Error Estimation

(i) We want to use a Taylor polynomial of degree for \( \cos x \) centered at 0 to estimate \( \cos 0.1 \). Find \( n \) such that the error in doing so is at most \( 10^{-6} \). Then carry out the approximation.

(ii) Use the 5\(^{th}\) degree Taylor polynomial for \( e^{3x} \) to estimate \( e^{0.3} \). Bound the error as well.

(iii) Find the second degree Taylor polynomial for \( e^t \) and use it to give an estimate for the integral \( \int_0^1 e^{x^2} \, dx \). Now suppose we instead used the 5\(^{th}\) degree Taylor polynomial \( p(t) \) for \( e^t \) to estimate that integral. Give an upper bound for the error

\[
\left| \int_0^1 e^{x^2} \, dx - \int_0^1 p(x^2) \, dx \right|
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