1. In this problem, we investigate the Geometric Series, one of the few series we actually know how to compute the sum of. Consider \( S_n = 1 + r + r^2 + \cdots + r^n \). Multiply both sides by \( r \), to find \( rS_n \). \( S_n \) is similar to \( S_n \)? Now look at \( S_n - rS_n \). Can you solve some equation for \( S_n \)? This is the finite geometric series, and this equation should hold true for all \( r \neq 1 \). Now, take \( \lim_{n \to \infty} S_n \), and recall that \( \lim_{n \to \infty} r^n = 0 \) if \( |r| < 1 \) and does not exist if \( |r| > 1 \). What’s that limit if \( r = \pm 1 \)? Does the series converge if \( r = \pm 1 \)?

2. Repeating decimals and Extra Terrestrials. In grade school or perhaps junior high, you may have been told that

\[
0.\overline{9} = 0.999999999\ldots = 1.
\]

Why might this be the case? On planet Zork, the native Zorkanians use the base 6 number system because they all have 6 fingers. This means that the counting numbers are \{1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, \ldots\} and decimals are given by \( 0.1 = \frac{1}{6} \), and \( 0.01 = \frac{1}{36} \). Can you find the numerical value of \( 0.12 \) in the Zorkanian number system?

3. In this game, we have a coin that lands heads with probability \( p \) and tails with probability \( q \). Note that \( p + q = 1 \) since there are only two possible outcomes, and if \( p = q = 1/2 \) we would have a ‘fair’ coin. The game is we flip the coin until we end up with a heads, a win. The number of flips required defines a random variable \( X \), whose probabilities are given by \( P(X = k) = p(1-p)^{k-1}, k = 1, 2, 3, \ldots \). You can think of this as the requirement that the first \( k-1 \) flips are tails, the \( (1-p)^{k-1} \) part, and the final flip must be heads, the \( p \) part.

(a) Verify that \( \sum_{k=1}^{\infty} P(X = k) = 1 \), which should be true. Why?

(b) We’ll now compute the mean (average) of this random variable.

For a discrete random variable as such, the mean is defined by \( E[X] = \sum_k kP(X = k) \). Start with the geometric series

\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n
\]

and differentiate both sides. Plug in an appropriate value for \( x \).
4. Detective Maclarurin (1698-1746) has hired you as a PI (private investigator) to help him recover his client’s function. His client knows what \( f(0), f'(0), f''(0), f'''(0), \ldots \) are, but for the life of him can’t figure out any other values! Detective Maclaurin assures you that in his lifetime of experience, we can recover the function by setting

\[
f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots
\]

and solving for the coefficients \( \{a_n\}_{n=1}^{\infty} \). For example, \( f(0) = a_0 + 0 + 0 + \cdots \), so we know that \( a_0 = f(0) \). Can you find all the other \( a_n \)'s?

How does this relate to the previous problem?

5. Explain the difference in meaning between the two expressions

\[
\int_1^{\infty} \frac{1}{x} - \frac{1}{x+1} \, dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x} \, dx - \int_1^{\infty} \frac{1}{x+1} \, dx.
\]

Discuss explicitly how values are assigned to each expression. Repeat with

\[
\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{1}{k+1}.
\]

6. Consider the function that is defined by \( f(n) = 1, f(n \pm 1/n^2) = 0 \) and for all \( x \)'s in between these points, is given by a straight line.

(a) Draw a graph of this function.
(b) Does \( \sum_{n=1}^{\infty} f(n) \) converge?
(c) Does \( \int_1^{\infty} f(x) \, dx \) converge?
(d) Did we just break the integral test?

Can you come up with a function \( f(x) \) such that \( \sum_{n=1}^{\infty} f(n) \) converges yet \( \int_1^{\infty} f(x) \, dx \) diverges?

7. Determine if the following series converge or diverge.

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<tbody>
<tr>
<td>(a)</td>
<td>( \sum_{n=1}^{\infty} \frac{n}{n+1} )</td>
<td>( b )</td>
<td>( \sum_{n=1}^{\infty} \frac{-10}{n} )</td>
<td>( c )</td>
</tr>
<tr>
<td>(d)</td>
<td>( \sum_{n=1}^{\infty} \frac{1}{1 + n^2} )</td>
<td>( e )</td>
<td>( \sum_{n=1}^{\infty} n \sin(\frac{1}{n}) )</td>
<td>( f )</td>
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8. **Old Exam Problems!** These problems are from an old midterm given by Dr. Malekpour. Find the indefinite integrals:

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<tbody>
<tr>
<td>(a)</td>
<td>( \int \frac{\cos(\sqrt{2}x)}{\sqrt{2}x} , dx )</td>
<td>( b )</td>
<td>( \int x \sqrt{3x + 2} , dx )</td>
<td>( c )</td>
</tr>
<tr>
<td>(c)</td>
<td>( \int \frac{dx}{x^2 - 5x + 4} )</td>
<td>( d )</td>
<td>( \int \sin(x) \cos(x)e^{\sin(x)} , dx )</td>
<td>( e )</td>
</tr>
<tr>
<td>(e)</td>
<td>( \frac{dx}{x^2 - 5x + 4} )</td>
<td>( f )</td>
<td>( \int \frac{-\sin(x)}{\cos^2(x) - 5\cos(x) + 4} , dx )</td>
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