1. Taylor’s remainder theorem states that given a function \( f(x) \) with infinitely many continuous derivatives, we can write this function as

\[
f(x) = T_n(x) + R_n(x).
\]

Provide a specific form for the two terms \( T_n(x) \) and \( R_n(x) \).

2. Determine whether or not the following series converge. Do they converge absolutely?

(a) \[
\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}
\]
(b) \[
\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}
\]
(c) \[
\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3+1}}
\]
(d) \[
\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}
\]
(e) \[
\sum_{n=2}^{\infty} \frac{\ln(n)}{\ln(\ln n)}
\]
(f) \[
\sum_{n=1}^{\infty} (1 + \frac{1}{n})^n
\]
(g) \[
\sum_{n=5}^{\infty} \frac{4^{n+1}}{5^n - 12^n + 11^n}
\]
If it converges, can you find its sum?

3. Evaluate \( \sum_{n=0}^{\infty} \int_{n}^{n+1} \frac{1}{1+x^2} \, dx \).

4. Badgers \( \heartsuit \) Taylor polynomials. The function \( f(x) = \sqrt{1-x^2} \) describes the top half of a circle.

(a) Show that the the only non-zero terms in the Taylor polynomial for \( f(x) \) are \( a_0, a_2, a_4, \ldots \).

(b) Since \( a_3 = 0 \), the two Taylor polynomials \( T_2(x) = T_3(x) \) are identical, so we can also say that \( R_2(x) = R_3(x) \). Which one is smaller?

(c) Compute

\[
\int_{0}^{0.1} T_3(x) \, dx.
\]

(d) The value you computed in part (c) should be close to \( \int f(x) \, dx \), since \( f(x) \approx T_3(x) \). Let’s make this more precise. Write down a formula for how much these two integral differ from each other.
(e) Can you provide an upper bound for how much these might differ? There are multiple correct answers, and the following inequality might be useful. If $g(x)$ is continuous, then

$$\left|\int_a^b g(x) \, dx\right| \leq \int_a^b |g(x)| \, dx.$$ 

5. There are a number of Taylor polynomials that keep showing up time and time again, so it’s useful to just have these memorized. Write down the Taylor series for the following functions.

(a) $e^x$  (b) $\cos(x)$  (c) $\sin(x)$  (d) $\frac{1}{1-x}$

6. Look at last Friday’s worksheet, or any of the practice problems from Professor Milewski’s previous exams.