1. This week we finished up the section on techniques of integration. Without looking at your notes/textbook, list as many techniques that you can think of. For each of these techniques can you come up with one integral to compute that uses that technique? For each pair of techniques, can you come up with an integral that uses both techniques? Take 5 minutes to think about this on your own, then discuss with your group.

2. Here we present a method that can be used for showing an integral doesn’t exist. If \( f(x) \) is continuous and \( f(x) \geq 0 \), then \( F(b) = \int_a^b f(x) \, dx \) is either bounded above, or it isn’t. If it’s bounded above, then \( \lim_{b \to \infty} F(b) \) exists, otherwise \( \int_a^\infty f(x) \, dx = \lim_{b \to \infty} F(b) = \infty \).

   (a) Suppose that \( 0 \leq g(x) \leq f(x) \) and that \( \int_a^\infty g(x) \, dx = \infty \). Show that \( \int_a^\infty f(x) \, dx = \infty \).

   (b) Can you interpret this result geometrically?

3. We define a set of functions \( L^p([a, b]) = \left\{ f(x) : \int_a^b |f(x)|^p \, dx < \infty \right\} \).

   (a) For what values of \( p \) does the function \( f(x) = \frac{1}{x} \) belong to \( L^p([1, \infty)) \)?

   (b) For what values of \( p \) does the function \( f(x) = \frac{1}{x} \) belong to \( L^p([0, 1]) \)?

   (c) Using your answer to part (a), can you come up with a function that belongs to \( L^p([1, \infty)) \) for all \( p > q \)?

   (d) Using your answer to part (b), can you come up with a function that belongs to \( L^p([0, 1]) \) for all \( p < q \)?

   (e) Using your answer from parts (c) and (d), can you come up with a function that belongs to \( L^p((\infty, \infty)) \) for all \( q < p < r \)?

4. This problem demonstrates a useful technique for demonstrating convergence of a function.

   (a) Show that if \( c > 0, x > 0 \), then \( \frac{1}{x+c} < x^{-1} \).

   (b) Show that if \( c > 0, x > 0 \) and \( r > 1 \), then \( \frac{1}{(x+c)^r} < x^{-r} \).

   (c) Use this result to determine if the following improper integrals converge or diverge.
i. \( \int_{1}^{\infty} \frac{1}{(x+1)^{2011}} \, dx \)

ii. \( \int_{1}^{\infty} \frac{0.5}{(4x+23)^{1/2}} \, dx \)

iii. \( \int_{5}^{\infty} \frac{x}{x-1} \, dx \). \text{Hint: what inequality do we use when} \( c < 0 \)?

5. One way to interpret \( \int_{-\infty}^{\infty} f(x) \, dx \) is \( \lim_{b \to \infty} \int_{-b}^{b} f(x) \, dx \). This is sometimes called the Cauchy principal part. Compute \( \lim_{b \to \infty} \int_{-b}^{b} x^3 \, dx \). Is something wrong? Repeat with \( f(x) = \frac{x}{x+1} \). Can we fix this?

6. \textbf{Introduction to Probability Theory} Improper integrals are very important for studying probability, and here we present an example.

(a) Let \( T \) denote the life time of a laptop battery. The probability that \( T \) will last \( a \) years or less is given by

\[
P(T \leq a) = \int_{0}^{a} f(x) \, dx, \quad \text{where} \quad f(x) = 2e^{-2x}.
\]

This is an example of a \textit{continuous random variable}, and \( f(x) \) is called the \textit{probability density function}. What is the probability that

i. the battery lasts 3 years or less?

ii. the battery lives at least 1 year?

iii. the battery dies in a finite amount of time?

iv. the battery lives forever?

(b) The \textit{Expected Value} or average value of a random variable is defined by

\[
E[T] = \int x f(x) \, dx
\]

If you were given a completely random battery, this would be your best guess to how long it would live. Compute \( E[T] \).

(c) A \textit{Normal} random variable has a density function given by \( f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \).

This is also known as a Gaussian (name after Gauss) and it’s graph is sometimes called a bell curve.

i. compute \( \int_{-\infty}^{\infty} f(x) \, dx \). \text{Hint: You should be able to do this without doing any integration.}

ii. Set \( \sigma = 1 \) and \( \mu = 0 \). Find \( E[X] \).

iii. Set \( \sigma = 1 \) and \( \mu = 1 \). Find \( E[X] \).

iv. Set \( \sigma = 1 \). Find \( E[X] \).