1. Interpreting integrals that involve ∞. Do this problem before looking at any other problem on the worksheet!

   (a) How do we interpret \( \int_{a}^{\infty} f(x) \, dx \) geometrically?
   (b) What is potentially wrong with what you said in part (a)?
   (c) How do we formalize what you said in part (a)?

2. Consider the integral \( \int_{-2}^{1} \frac{1}{x^2} \, dx \). Despite Dina’s protests, David says the way to do this is

   \[
   \int_{-2}^{1} \frac{1}{x^2} \, dx = - \left. \frac{1}{x} \right|_{-2}^{1} = - \left( \frac{1}{2} - (-1) \right) = - \frac{3}{2}.
   \]

   Why is the numerical value for the answer a big red flag? What was Dina protesting? How can we fix this?

3. **To paradox or not to paradox. That is the question.** Consider the function \( f(x) = \frac{1}{x} \). Find the area under the curve of \( f(x) \) from \( x = 1 \) to \( x = \infty \). Rotate this function around the \( x \)-axis to form a solid. Find the volume of this solid. Is this a paradox?

4. Dina pointed out to David that he didn’t record her method correctly on the last worksheet. She claims that the correct PFD for \( \frac{x^2+1}{x(x+1)^2} \) is \( \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \), and not \( \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x(x+1)} \). Show that her correct method works, and that the second one does not by solving for each coefficient, then adding the resulting fractions.

   Trent still asserts that \( \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \) still works. Can you come up with at least one advantage each method has over the other method?

5. The ‘Heaviside’ method can be used for monomial terms in a partial fraction decomposition. Find \( A \) if \( \frac{p(x)}{q(x)(x-r)} = \frac{A}{x-r} + \frac{p_1(x)}{q_2(x)} \). Use this method to decompose the following. In each case, indicate what you’re using for \( p, q \) and \( r \).

   (a) \( \frac{x}{(x+1)(x-2)} \)
   (b) \( \frac{x^2+1}{x(x-1)(x+5)} \)

   How does this method compare with the usual methods? What if any limitations does this method have?
6. The formal definition for $\int_a^\infty f(x) \, dx$ is $\lim_{b \to \infty} \int_a^b f(x) \, dx$. One very important question in mathematics is to determine criteria under which this limit exists, and here we present one such criteria. One fundamental property of the real numbers is the following (oftentimes called the least upper bound property). If $f(x)$ is an increasing function, and $f(x) \leq M$ for some real number $M$, then $\lim_{x \to \infty} f(x)$ exists. This is oftentimes taken as an axiom.

(a) Suppose $f(x) \geq 0$ for all $x$. Show that $F(b) = \int_a^b f(x) \, dx$ is an increasing function.

(b) Suppose further that $f(x) \leq g(x)$ and that $\int_a^\infty g(x) \, dx < \infty$. Show that $F(x)$ is bounded above.

(c) Use the least upper bound property to conclude that $\int_a^\infty f(x) \, dx$ exists.

(d) Use this result to show that $\int_1^\infty \frac{\sin(\alpha x)}{x^2} \, dx$ exists for any $\alpha$.

7. We define a set of functions $L^p([a,b]) = \{ f(x) : \int_a^b |f(x)|^p \, dx < \infty \}$, where $a$ could possibly be $-\infty$, $b$ could possibly be $\infty$ and $p > 0$. The membership criteria for a function is simply to take the absolute value of it, raise it to the $p^{th}$ power, integrate it and see if the result is finite.

(a) For what values of $p$ does the function $f(x) = \frac{1}{x}$ belong to $L^p([1, \infty))$?

(b) For what values of $p$ does the function $f(x) = \frac{1}{x}$ belong to $L^p([0, 1])$?

(c) Using your answer to part (a), can you come up with a function that belongs to $L^p([1, \infty))$ for all $p > q$?

(d) Using your answer to part (b), can you come up with a function that belongs to $L^p([0, 1])$ for all $p < q$?

(e) Using your answer from parts (c) and (d), can you come up with a function that belongs to $L^p((-\infty, \infty))$ for all $q < p < r$?

8. Determine if $\int_1^\infty \frac{\cos(x)}{x} \, dx$ converges. *Hint:* Problem 6 might be useful.