1. Integrands oftentimes involve $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{x^2 + a^2}$. For each of these cases, can you come up with a useful substitution that may help to simplify an integrand? It may be helpful to start with your favorite trig id: $\sin^2(\theta) + \cos^2(\theta) = 1$. Use what you deduced to evaluate the following:

(a) $\int \frac{1}{\sqrt{1-x^2}} \, dx$
(b) $\int \frac{1}{\sqrt{4-x^2}} \, dx$
(c) $\int \frac{3}{\sqrt{x^2+9}} \, dx$
(d) $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x} \, dx$
(e) $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

2. Suppose $f(x)$ is a function that satisfies $\int_0^1 f(x) f''(x) \, dx = 0$. Show that if $f(0) = f'(1) = 0$, then $f(x) = 0$ for all $x \in [0,1]$. Hint: If $g(x) \geq 0$ is a continuous function and satisfies $\int_a^b g(x) \, dx = 0$, can you say anything about $g(x)$? Is it important to assume that $g$ is continuous?

3. Dina says that the correct PFD for $\frac{1}{x(x+1)^2}$ is $\frac{4}{x} + \frac{B}{x+1} + \frac{C}{x+1}$ because $x+1$ is a repeated root. Trent claims that you only need to use one degree lower than $(x+1)^2$, so he sets it up like $\frac{A}{x} + \frac{Bx+C}{(x+1)^2}$. Will both setups work? Why?

4. There are at least two schools of thought for doing partial fractions. The first involves equating coefficients, and I’ll call the second method the ‘plug and chug’ method. In both cases, you start out by multiplying both sides by the denominator of the left hand side, then proceeding with a method. Use the method of equating coefficients to integrate the following:

(a) $\int \frac{2x}{(x+1)(x+2)} \, dx$
(b) $\int \frac{x+2}{x^2+9} \, dx$
(c) $\int \frac{x}{(x^2+1)(x-1)} \, dx$
(d) $\int \frac{x^2-2x+1}{(x+1)(x-2)} \, dx$
(e) $\int \frac{x^3-2x+1}{x^3-2x^2-x-2} \, dx$

(f) $\int \frac{z-1}{z^3(z-1)(z^2+1)} \, dx$

Recall that if $r = p/q$ is a rational root of the polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $p$ divides $a_0$ and $q$ divides $a_n$. This might be useful when factoring some of these. Repeat these problems using the other method. Which do you prefer? When is one method easier than the other? Can you combine these methods?

5. The ‘Heaviside’ method can be used for monomial terms in a partial fraction decomposition. Find $A$ if $\frac{p(x)}{q(x)(x-r)} = \frac{A}{x-r} + \frac{p_1(x)}{q_2(x)}$. Use this method to decompose the following. In each case, indicate what you’re using for $p, q$ and $r$.

(a) $\frac{x}{(x+1)(x-2)}$

(b) $\frac{x^2+1}{x(x-1)(x+5)}$

How does this method compare with the methods in problem 4?

6. (a) Set $I_n = \int \cos^n x \, dx$. Show that $I_n = \frac{1}{n} \cos^{n-1} x \sin + \frac{n-1}{n} \int \sin^{n-2} x \, dx$.
   What values of $n$ does this hold for? What’s $I_1$ and $I_3$? Can you find $I_3$ using a different method?

(b) If we set $I_n = \int (\ln x)^n \, dx$, can you find a formula for $I_n$ in terms of $I_{n-1}$? Find $I_1$ by hand, and use that to find $I_2$. 