4.2. **Vector Space** $\mathbb{R}^n$ and Subspaces. Vector Spaces are (in the abstract sense) sets of elements, called vectors, that are endowed with a certain group of properties. You can add two vectors and multiply vectors by scalars. They satisfy the usual commutative, associative and distributive laws you would expect. These properties are listed on page 236 of your textbook.

Many problems we encounter can actually be viewed as vector spaces. In fact, this may come as a surprise to you, but functions are vectors. In fact, solutions to linear differential equations can also be thought of as a linear subspace of a certain class of functions. This is the motivation for studying vectors in the abstract sense.

If we have a vector space $V$, and we have a subset $W \subset V$, a natural question to ask is whether or not $W$ itself forms a vector space. This means it needs to satisfy all the properties of a vector space from above! The bad news is this is quite a long list, however the good news is we don’t have to check every property on the list, because most of them are inherited from the original vector space $V$. In short, in order to see if $W$ is a vector space, we need only check if $W$ passes the following test.

**Theorem 1.** If $V$ is a vector space and $W \subset V$ is a non-empty subset, then $W$ itself is a vector space provided it satisfies the following two conditions:

- **a. Additive Closure** If $\vec{a} \in W$ and $\vec{b} \in W$, then $\vec{a} + \vec{b} \in W$.
- **b. Multiplicative Closure** If $\lambda \in \mathbb{R}$ and $\vec{a} \in W$, then $\lambda \vec{a} \in W$.

Note that these are two properties that are on the long laundry list of properties we require for a set to be a vector space.

**Example** Consider $W := \{ \vec{a} = (x, y) \in \mathbb{R}^2 : x = 2y \}$. Since $W \subset \mathbb{R}^2$, we may be interested if $W$ itself forms a vector space. To answer this question we need only check two items:

- **a. Additive Closure:** An arbitrary element in $W$ can be described by $(2y, y)$ where $y \in \mathbb{R}$. Let $(2y, y), (2z, z) \in W$. Then $(2y, y) + (2z, z) = (2y + 2z, y + z) \in W$ since $2y + 2z = 2(y + z)$.
- **b. Multiplicative Closure:** We need to check if $\lambda \in \mathbb{R}$ and $\vec{a} \in W$, then $\lambda \vec{a} \in W$.

Again, an arbitrary element in $W$ can be described by $(2y, y)$ where $y \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$ and $(2y, y) \in W$. Then $\lambda(2y, y) = (2\lambda y, \lambda y) \in W$ since the first coordinate is exactly twice the second element.

Note: it is possible to write this set as the kernel of a matrix. In fact, you can check that $W = \ker(A)$, where $A_{1 \times 2} = \begin{bmatrix} -2 & 1 \end{bmatrix}$. We actually have a theorem that says the kernel of any matrix is indeed a linear subspace.
Example Consider \( W := \{ \vec{a} \in \mathbb{R}^3 : \ z \geq 0 \} \). In order for this to be a linear subspace of \( \mathbb{R}^3 \), it needs to pass two tests. In fact, this set passes the additive closure test, but it doesn’t pass multiplicative closure! For example, \( (0,0,5) \in W \), but \( (-1) \cdot (0,0,5) = (0,0,-5) \not\in W \).

Definition. If \( A_{m \times n} \) is a matrix, we define \( \ker(A) := \{ x \in \mathbb{R}^n : \ Ax = 0 \} \). This is also called the nullspace of \( A \).

Note that \( \ker(A) \) lives in \( \mathbb{R}^n \).

Definition. If \( A_{m \times n} \) is a matrix, we define \( \text{Image}(A) := \{ y \in \mathbb{R}^m : \ Ax = y \text{ for some } x \in \mathbb{R}^n \} \). This is also called the range of \( A \).

Note that \( \text{Image}(A) \) lives in \( \mathbb{R}^m \).

Theorem 2. If \( A_{m \times n} \) is a matrix, then \( \ker(A) \) is a linear subspace of \( \mathbb{R}^n \) and \( \text{Image}(A) \) is a linear subspace of \( \mathbb{R}^m \).

4.3. Linear Combinations and Independence of Vectors. If we have a collection of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \), we can form many vectors by taking linear combinations of these vectors. We call this space the span of a collection of vectors, and we have the following theorem:

Theorem 3. If \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is a collection of vectors in some vector space \( V \), then

\[ W := \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} := \{ \vec{w} : \vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k, \text{ for some scalars } c_i \} \]

is a linear subspace of \( V \).

For a concrete example, we can take two vectors \( \vec{v}_1 = (1,1,0) \) and \( \vec{v}_2 = (1,0,0) \) which both lie in \( \mathbb{R}^3 \). Then the set \( W = \text{span}\{(1,1,0),(1,0,0)\} \) describes a plane that lives in \( \mathbb{R}^3 \). This set is a linear subspace by this previous theorem. In fact, we can be a bit more descriptive and write \( W = \{(x,y,z) \in \mathbb{R}^3 : \ z = 0 \} \).

If we continue with this example, it is possible to write \( W \) in many other ways. In fact, we could have written

\[ W = \text{span}\{(1,1,0),(1,0,0),(-5,1,0)\} = \text{span}\{(-10,1,0),(2,1,0)\} \]

These examples illustrate the fact that our choice of vectors need not be unique. What is unique, is the least number of vectors that are required to describe the set. In fact this is so important we give it a name, and call it the dimension of a vector space. This is the content of section 4.4. In our example, \( \dim(W) = 2 \), but right now we don’t have enough tools to show this.

In order to make this statement ‘least’, precise, we need to introduce definition.

Definition. Vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) are said to be linearly independent if whenever

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = 0 \]

for some scalars \( c_i \), it must follow that \( c_i = 0 \) for each \( i \).

OK, so definitions are all fine and good, but how do we check if vectors are linearly independent? The nice thing about this definition is it always boils down to solving a linear system.

Example As a concrete example, let’s check if vectors \( \{ \vec{v}_1, \vec{v}_2 \} \) linearly independent where \( \vec{v}_1 = (4,-2,6,-4) \) and \( \vec{v}_2 = (2,6,-1,4) \).
We need to solve the problem
\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 = 0. \]
This reduces to asking what are the solutions to
\[
\begin{bmatrix}
4 & 2 \\
-2 & 6 \\
6 & -1 \\
-4 & 4
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]
We can write this problem as a matrix equation \( A\vec{c} = \vec{0} \) where
\[
A = \begin{bmatrix}
4 & 2 \\
-2 & 6 \\
6 & -1 \\
-4 & 4
\end{bmatrix}, \quad \vec{c} = \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
\]
and solve this using Gaussian elimination.
\[
\begin{bmatrix}
4 & 2 & 0 \\
-2 & 6 & 0 \\
6 & -1 & 0 \\
-4 & 4 & 0
\end{bmatrix}
\rightarrow \text{row ops}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Thus \( c_1 = c_2 = 0 \) is the only solution to this problem, and so these two vectors are linearly independent.

To demonstrate that a collection of vectors are not linearly independant, it suffices to find a non-trivial combination of these vectors and show they sum to \( \vec{0} \). For example, see example 6 in the textbook.