1. Let \( n \in \mathbb{N} \) and \( I = \{1, 2, \ldots, n\} \). For \( i \in I \), define \( A_i = [(i-1)/n, i/n] \). Identify each of the following sets by writing it as an interval or a union of two intervals.

(a) \( \bigcup_{i \in I} A_i \)
(b) \( \bigcap_{i \in I} A_i \)

2. Consider the set \( S = \{0, \Box\} \).

(a) List the elements of \( \mathcal{P}(S) \).
(b) List the elements of \( \mathcal{P}(\mathcal{P}(S)) \).
(c) Find a partition of \( \mathcal{P}(\mathcal{P}(S)) \) into 3 sets.
(d) Is it possible to find a partition of \( S \) into 3 sets? Explain.

3. Prove the statements appearing in (a)-(b), and answer the prompt in (c)-(d). The symbol \( \equiv \) denotes congruence modulo \( n \), where \( n \in \mathbb{Z} \) such that \( n \geq 2 \).

(a) For all \( a, b \in \mathbb{Z} \), if \( a \equiv b \), then \( b \equiv a \).
(b) For all sets \( a, b, c \in \mathbb{Z} \), if \( a \equiv b \) and \( b \equiv c \), then \( a \equiv c \).
(c) State the negation of each of the statements (a)-(b) above. Determine if the negation is true or false. Provide a counterexample for any false statement.
(d) Let \( a, b, c \in \mathbb{Z} \), and consider the conditional statement

\[ P: \text{If } a \equiv b \text{ and } b \equiv c, \text{ then } a \equiv c. \]

State the inverse, contrapositive and converse of statement \( P \). Determine whether each of these is true or false.

4. Negate the following.

(a) \( \forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \text{ such that } m \cdot n = 1. \)
(b) \( \exists x \in \mathbb{Q} \text{ such that } \forall y \in \mathbb{Q}, x \cdot y = y. \)

Rewrite the statements in (a) and (b) without the use of the symbols \( \forall, \exists \), and state whether each is a true or a false statement. If it is a true statement, prove it. If it is a false statement, provide a counterexample.

5. Construct a truth table to show that the contrapositive of \( A \Rightarrow B \) is equivalent to \( A \Rightarrow B \).

6. Prove the following statement.

\[ \forall a \in \mathbb{R} \exists x \in \mathbb{R}, \text{ such that } 3x - 1 = a. \]

7. Let \( E \) denote the set of even integers, \( x \in \mathbb{Z} \), and \( A(x) \) be the following open sentence.

\[ A(x) : \text{“} x \in E \Rightarrow \exists k \in \mathbb{Z} \text{ such that } x = 2k \text{”} \]

(a) Write the inverse of \( A(x) \).
(b) Write the converse of \( A(x) \).
(c) Write the contrapositive of \( A(x) \).
(d) Is \( A(x) \) true for all \( x \in \mathbb{Z} \)? What about its converse? In this case, how would you restate it using necessary/ sufficient/

necessary and sufficient?

8. Let \( A = \{ x \in \mathbb{Z} | x = 6k, k \in \mathbb{Z} \} \),
\( B = \{ x \in \mathbb{Z} | x = 2k, k \in \mathbb{Z} \} \),
\( C = \{ x \in \mathbb{Z} | x = 3k, k \in \mathbb{Z} \} \). Prove the following statement.

\[ x \in A \iff (\exists y \in B \text{ and } \exists z \in C \text{ such that } x = yz) \]

9. Construct a truth table to show that \( A \Rightarrow B \) is equivalent to the statement: \( \neg(\neg A \lor \neg B) \) or \( B \).

10. Let \( a, b, c \in \mathbb{R} \), and consider the following open sentence:

\[ P(a, b, c) : \text{A necessary condition for the equation } ax^2 + bx + c = 0 \text{ to have a solution is: } a \neq 0 \text{ and } b^2 - 4ac \geq 0. \]

(a) Rephrase \( P(a, b, c) \) as an if-then implication; explicitly write all relevant quantifiers.
(b) Write the contrapositive.
(c) Write the converse.
(d) Write the inverse.
(e) Write the negation of \( P(a, b, c) \) (simplified by moving the \( \neg \) as far into the statement as possible).
(f) Which of the above statements (a)-(d) are equivalent to each other (for all \( a, b, c \in \mathbb{R} \)?)
(g) The statement \( \forall a, b, c \in \mathbb{R}, P(a, b, c) \) is false. Disprove it (prove the negation).

11. Let \( x, y \in \mathbb{R} \). Prove that \( (x + y)^2 = x^2 + y^2 \) if and only if \( xy = 0 \).

12. Let \( n \in \mathbb{Z} \). Prove that \( n \) is odd if and only if \( n + 7 \) is even.

13. Let \( n \in \mathbb{Z} \). Prove that \( n \) is odd if and only if \( n^2 \) is odd.

14. Let \( a \in (0, \infty) \). Prove that a rectangle with perimeter \( 4a \) is a square if and only if its area is \( a^2 \).
15. Define the Euclidean norm of 
\( x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \) by 
\( ||x|| = \sqrt{x_1^2 + ... + x_n^2} \).
Prove that \( ||x|| = 0 \) if and only if 
\( (x_1, x_2, ..., x_n) = (0, 0, ..., 0) \).

16. Let \( x \in \mathbb{Z} \).
   (a) Prove that \( x^2 + x \) is even.
   (b) Assume \( x \neq 0 \). Prove that \( (x^2 + x)/2 \) is divisible by \( x \) if and only if \( x \) is odd.
   (c) Assume \( x + 1 \neq 0 \). Prove that \( (x^2 + x)/2 \) is divisible by \( x + 1 \) if and only if \( x \) is even.

17. Show that if \( x^2 - 3x + 2 < 0 \), then \( 1 < x < 2 \).

18. Let \( a, b, c, d \in \mathbb{Z} \) with \( a \) and \( b \) nonzero. Prove that 
   if \( ab \nmid cd \), then \( a \nmid c \) or \( b \nmid d \).

19. Prove that for any two sets \( A \) and \( B \), 
\( (A \cup B) = A \cap B \).

20. Prove that for any two sets \( A \) and \( B \), 
\( (A \cup B) - (A \cap B) = (A - B) \cup (B - A) \).

21. Prove that for any sets \( A, B \) and \( C \), 
\( A \times (B \cup C) = (A \times B) \cup (A \times C) \).

22. Prove that if \( n \mid a \) then \( n \mid a + b \iff n \mid b \)

23. Let \( A = \{ x \in \mathbb{R} : |x - 1| \leq 1 \} \) and let \( B \) be the interval \( [0, 3] \). Give a geometric description of 
\( A \times B \) as a subset of \( \mathbb{R} \times \mathbb{R} \). Can you conclude that 
\( \overline{A \times B} = \overline{A} \times \overline{B} \)? (Here \( \overline{A \times B} \) denotes the complement of \( A \times B \) in \( \mathbb{R} \times \mathbb{R} \), while \( \overline{A} \) and \( \overline{B} \) denote the complements of \( A \) and \( B \) in \( \mathbb{R} \).)