Directions:

- Include a cover page.
- Do not hand in scratch work. The final version of your solution to each problem should be collocated and stapled.
- Always label every plot (title, x-label, y-label, and legend).
- Use the subplot command from MATLAB (or matplotlib) when comparing 2 or more plots to make comparison easier and to save paper.

1. Compute the first 5 terms in the Taylor series (constant, linear, quadratic, cubic, and quartic pieces) for the following functions about the given point:

   (a) \( f(x) = \sin(x) \), about the point \( a = \pi/4 \).
   (b) \( f(x) = x/(1+x) \), about the point \( a = 0 \). \textbf{Hint}: you may want to start with \( 1/(1+x) \).
   (c) \( f(x) = e^{\cos(x)} \), about the point \( a = 0 \).
   (d) \( f(x) = 1 - x - 2x^2 + x^3 \), about the point \( a = 1 \).

2. Using the results from Problem 1(c), make a single MATLAB (or matplotlib) plot which contains all of the following:

   (a) A graph of \( f(x) = e^{\cos(x)} \) versus \( x \) for \( x \in (-3, 3) \).
   (b) A graph of \( P_2(x) \).
   (c) A graph of \( P_4(x) \).
   (d) A title, x-axis label, y-axis label, and a legend.

3. Determine which one of the following sequences converges to 1 faster (clearly explain your reasoning):

   \[ \lim_{x \to 0} \frac{\sin(x^2)}{x^2} \quad \text{and} \quad \lim_{x \to 0} \frac{\sin^2(x)}{x^2}. \]

4. Compute each of the following limits and determine the corresponding rate of convergence:

   (a) \( \lim_{x \to 0} \frac{e^x - 1}{x} \)
   (b) \( \lim_{x \to 0} \frac{\sin(x)}{x} \)
   (c) \( \lim_{x \to 0} \frac{\cos(x) - 1 + x^2/2 - x^4/24}{x^6} \)

---

1Yes, doing math will require a lot of scratch work! You will rarely be able to correctly answer each question on your first try. For each problem, figure out what the solution is, and then write-up a final draft that you hand in.
5. Consider the function \( f(x) = e^x \).

(a) Derive the \( n^{th} \) Taylor polynomial \( P_n(x) \) as well as the remainder term \( R_n(x) \) for the function \( f(x) \), expanded about the point \( x = 0 \).

(b) Using the remainder term from part (a), determine the value of \( n \) needed to guarantee that \( |P_n(-1) - f(-1)| < 10^{-5} \).

(c) Compute the actual error, \( err(-1) = |P_n(-1) - f(-1)| \).

6. Suppose theory indicates that a sequence \( \{p_n\} \) converges to \( p \) with order 1.5. Explain how you would numerically verify this order of convergence. In particular, demonstrate your method by constructing a table: select an asymptotic error constant \( \lambda = 0.5 \), and an initial approximation that satisfies \( |e_1| = 1 \).

7. Consider the recursive sequence \( \{x_n\} \) defined by

\[
x_{n+1} = \frac{x_n^3 + 3x_n a}{3x_n^2 + a}.
\]

(a) Suppose \( x = \lim_{n \to \infty} x_n \) exists. Show that \( x = 0 \) or \( x = \pm \sqrt{a} \).

\text{Hint}: If this limit exists, then \( x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1} \).

(b) Show that the convergence of this sequence toward \( \sqrt{a} \) is third order.

(c) What is the asymptotic error constant?

(d) Numerically verify your results by constructing a table similar to the proposed table in the previous problem. For the numerics only, use \( a = 1 \) and a starting value of \( x_0 = 131 \). You will only get in a few iterations before hitting machine precision.

8. Consider the functions \( f(x) = 1/(1 - x) \) and \( g(x) = 1/(1 + x) \).

(a) Obtain the infinite Taylor series representation of \( f(x) \) about the point \( x = 0 \).

(b) Obtain the infinite Taylor series representation of \( g(x) \) about the point \( x = 0 \).

\text{Hint}: you may reuse what you computed in Part (a).

(c) Obtain the infinite Taylor series representation for \( \ln(1 + x) \) by identifying this function as

\[
\ln(1 + x) = \int_0^x \frac{1}{1 + t} \, dt.
\]