Exam 1

Math 133
September 18th, 2012

Name: Key

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Your Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Q2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Q3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Q4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Q5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Q6</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Read all of the following information before starting the exam:

- Show all work, clearly and in order. “Answers” without justification will receive zero credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- All exams at Michigan State University are governed by our Academic Integrity Policy: https://www.msu.edu/~ombud/academic-integrity/index.html. Simply put, don’t cheat. There are serious consequences.
- Wait until instructed to begin exam to start. Good luck!
Problem 1 (20 points) Compute $\frac{dy}{dx}$ for each of the following:

(a) $y = e^{-\pi x}$
\[
\frac{dy}{dx} = e^{-\pi x} \frac{d}{dx} (-\pi x) = -\pi e^{-\pi x}
\]

(b) $y = x^{-\pi}$
\[
\frac{dy}{dx} = (-\pi) x^{-\pi - 1} = -\pi x
\]

(c) $y = 2 \ln \left( \frac{x}{a} \right)$
\[
= 2 \left( \ln x - \ln \left( \frac{1}{a} \right) \right) = 2 \ln x - 2 \ln a
\]
\[
\frac{dy}{dx} = \frac{-2}{x}
\]

(d) $y = \ln(2 \sin(\ln x))$
\[
\frac{dy}{dx} = \frac{1}{\sin \ln x} \left( \sin \ln x \right)'
\]
\[
= \frac{1}{\sin(\ln x)} \cos(\ln x) \left( \ln x \right)'
\]
\[
= \frac{1}{\sin(\ln x)} \cos(\ln x) \frac{1}{x}
\]
Problem 2 (10 points) The length of the curve which travels along the graph of \( y = e^{5x} \ln x \) from the point \((1, 0)\) to the point \((e, e^{5e})\) is found by evaluating an appropriate integral. Set up, but do not evaluate this integral.

\[
\frac{dy}{dx} = 5e^{5x} \ln x + \frac{e^{5x}}{x}.
\]

\[
L = \sqrt{1 + (5e^{5x} \ln x + \frac{e^{5x}}{x})^2} \, dx
\]

Problem 3 (10 points)
(a) Given the function \( f(x) = 3(x - 4)^2 + 1 \) whose domain is \( x \leq 4 \), find \( f^{-1}(x) \).

\[
y = 3(x - 4)^2 + 1\]
\[
y - 1 = 3(x - 4)^2
\]
\[
\frac{y - 1}{3} = (x - 4)^2
\]
\[
\sqrt{\frac{y - 1}{3}} = x - 4
\]
\[
x = 4 + \sqrt{\frac{y - 1}{3}}
\]

Swapping \( x \leftrightarrow y \):

\[
f^{-1}(x) = 4 + \sqrt{\frac{x - 1}{3}}
\]

(b) Compute \((f^{-1})'(4)\).

\[
b = 4.
\]

Note \( f(3) = 3(3 - 1)^2 + 1 = 4 \).

\[
(f^{-1})'(4) = \frac{1}{f'(3)}
\]

\[
(f'(x)) = 6(x - 4)
\]
\[
f'(3) = -6
\]

\[
(f^{-1})'(3) = -\frac{1}{6}
\]
Problem 4 (20 points) A force of 3 lbs is required to stretch an ideal spring 1 ft from its resting, equilibrium position.

(a) Compute the spring constant, $k$. What units, if any, does $k$ have?

\[ F = kx \]

\[ 3 = k(1) \quad \text{so} \quad k = 3 \, \frac{\text{lbs}}{\text{ft}} \]

(b) How much work is required to stretch the spring 2 feet from its equilibrium position?

\[ W = \int F \, dx = \int 3 \, dx = \frac{3}{2} x^2 \bigg|_0^2 \]

\[ = \frac{3}{2} (2)^2 = 6 \, \text{ft-lbs} \]
Problem 5 (20 points) Evaluate each of the following integrals:

(a) \[
\int_{\ln(1/2)}^{\ln(2)} e^x \sin(2e^x - 1) \, dx
\]

\[
= \int e^x \sin(u) \, \frac{du}{2e^x} = \frac{1}{2} \int \sin(u) \, du
\]

\[
= -\frac{1}{2} \cos u \bigg|_{x=1/2}^{x=2} = -\frac{1}{2} \cos u \bigg|_{u=0}^{u=3}
\]

\[
x = \ln(1/2);
\]

\[
u = 2e - 1 = 1 - 1 = 0
\]

\[
x = \ln(2);
\]

\[
u = 2e^2 - 1 = 84 - 1 = 3
\]

\[
\int_{2 + e^{x^3}}^{2 + e^{x^3}} 3x^2 \, dx
\]

\[
= \int \frac{du}{u} = \ln |u| + C
\]

\[
= \ln (2 + e^{x^3}) + C
\]

Can remove 1-1 signs because \(2 + e^{x^3} \geq 0\).
Problem 6 (20 points) The region in the first quadrant bounded below by \( y = 4x^2 \) and above by \( y = 4 \) is revolved about the \( y \)-axis.

(a) Find the volume of the resulting solid.

\[
V = \int 2\pi r^2 \, dy = \int_0^4 \frac{y^2}{4} \, dy = \frac{16}{3}
\]

(b) Now, suppose that the solid represents the interior of a tank, and this tank is filled with water with constant density \( \rho = 15 \, N/m^3 \). Set up, but do not evaluate an integral that describes the amount of work necessary to pump all of the water to the surface of the tank at \( y = 4 \).

\[
W = \int \rho \cdot A \cdot (\text{height}^+) \, dy = 15 \int_0^4 \left( \frac{y^2}{4} \right) (4-y) \, dy
\]
**Bonus:** (5 points - all or nothing)
Show that the function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(x) = 3x^{11} + 2x^5 + 2x \) has an inverse.

\[ f' = 33x^{10} + 10x^4 + 2 \geq 2, \]
\[ \geq 0 \quad \geq 0 \]

\[ \therefore \quad f \text{ is always increasing} \]

\[ \therefore \quad f \text{ has an inverse.} \]
(this page intentionally left blank)