The differential equation we’re interested in studying is
\[(1) \quad y' = f(t, y), \quad y(t_0) = y_0.\]

Many first order differential equations fall under this category and the following method is a new method for solving this differential equation. The first idea is to transform the DE into an integral equation, and then apply a new method to the integral equation.

We first do a change of variables to transform the initial conditions to the origin. Explicitly, you can define \(w = y - y_0\) and \(x = t - t_0\). With a new \(f\), the differential equation we’ll study is given by
\[(2) \quad y' = f(t, y), \quad y(0) = 0.\]

Note: it’s not necessary to do this substitution, but it makes life a lot easier if we do.

Now, we integrate equation (2) from \(s = 0\) to \(s = t\) to obtain
\[
\int_{s=0}^{t} y'(s) \, ds = \int_{s=0}^{t} f(s, y(s)) \, ds.
\]

Applying the fundamental theorem of calculus, we have
\[
\int_{s=0}^{t} y'(s) \, ds = y(t) - y(0) = y(t).
\]

Hence we reduced the differential equation to an equivalent integral equation given by
\[(3) \quad y(t) = \int_{s=0}^{t} f(s, y(s)) \, ds.\]

Even though this looks like it’s ‘solved’, it really isn’t because the function \(y\) is buried inside the integrand. To solve this, we attempt to use the following algorithm, known as Picard Iteration:

1. Choose an initial guess, \(y_0(t)\) to equation (3).
2. For \(n = 1, 2, 3, \ldots\), set \(y_{n+1}(t) = \int_{s=0}^{t} f(s, y_n(s)) \, ds\)

Why does this make sense? If you take limits of both sides, and note that \(y(t) = \lim_n y_{n+1} = \lim_n y_n\), then \(y(t)\) is a solution to the integral equation, and hence a solution to the differential equation. The next question you should ask is under what hypotheses on \(f\) does this limit exist? It turns out that sufficient hypotheses are the \(f\) and \(f_y\) be continuous at \((0, 0)\). These are exactly the hypotheses given in your existence/uniqueness theorem 2.

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Note: If we stop this algorithm at a finite value of $n$, we expect $y_n(t)$ to be a very good approximate solution to the differential equation. This makes this method of iteration an extremely powerful tool for solving differential equations!

For a concrete example, I’ll show you how to solve problem #3 from section 2 – 8.

Use the method of picard iteration with an initial guess $y_0(t) = 0$ to solve:

$$y' = 2(y + 1), \quad y(0) = 0.$$

Note that the initial condition is at the origin, so we just apply the iteration to this differential equation.

$$y_1(t) = \int_{s=0}^{t} f(s, y_0(s)) \, ds = \int_{s=0}^{t} 2(y_0(s) + 1) \, ds = \int_{s=0}^{t} 2 \, ds = 2t.$$

Hence, we have the first guess is $y_1(t) = 2t$. Next, we iterate once more to get $y_2$:

$$y_2(t) = \int_{s=0}^{t} f(s, y_1(s)) \, ds = \int_{s=0}^{t} 2(y_1(s) + 1) \, ds = \int_{s=0}^{t} 2(2s + 1) \, ds = \frac{2^2}{2!}t^2 + 2t.$$

Hence, we have the second guess $y_2(t) = \frac{2^2}{2!}t^2 + 2t$. Iterate again to get $y_3$:

$$y_3(t) = \int_{s=0}^{t} 2(y_2(s) + 1) \, ds = \int_{s=0}^{t} 2\left(\frac{2^2}{2!}s^2 + 2s + 1\right) \, ds = \frac{(2t)^3}{3!} + \frac{(2t)^2}{2!} + 2t.$$

It looks like the pattern is

$$y_n(t) = \sum_{k=1}^{n} \frac{(2t)^k}{k!}$$

and hence the exact solution is given by

$$y(t) = \lim_{n \to \infty} y_n(t) = \sum_{k=1}^{\infty} \frac{(2t)^k}{k!} = \sum_{k=0}^{n} \frac{(2t)^k}{k!} - 1 = e^{2t} - 1.$$

If you plug this into the differential equation, you’ll see we hit this one on the money. To demonstrate this solution actually works, below is a graph of $y_5(t)$, $y_{15}(t)$ and $y(t)$, the exact solution.