PICARD ITERATION

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The differential equation we're interested in studying is

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$

Many first order differential equations fall under this category and the following method is a new method for solving this differential equation. The first idea is to transform the DE into an *integral equation*, and then apply a new method to the integral equation.

We first do a change of variables to transform the initial conditions to the origin. Explicitly, you can define $w = y - y_0$ and $x = t - t_0$. With a new f, the differential equation we'll study is given by

$$y' = f(t, y), \qquad y(0) = 0. \tag{2}$$

Note: it's not necessary to do this substitution, but it makes life a lot easier if we do.

Now, we integrate equation (2) from $s = 0$ to $s = t$ to obtain

$$\int_{s=0}^{t} y'(s)\ ds = \int_{s=0}^{t} f(s, y(s))\ ds.$$

Applying the fundamental theorem of calculus, we have

$$\int_{s=0}^{t} y'(s)\ ds = y(t) - y(0) = y(t).$$

Hence we reduced the *differential equation* to an equivalent *integral equation* given by

$$y(t) = \int_{s=0}^{t} f(s, y(s))\ ds. \tag{3}$$

Even though this looks like it's 'solved', it really isn't because the function y is buried inside the integrand. To solve this, we attempt to use the following algorithm, known as Picard Iteration:

(1) Choose an initial guess, $y_0(t)$ to equation (3).
(2) For $n = 1, 2, 3, \ldots$, set $y_{n+1}(t) = \int_{s=0}^{t} f(s, y_n(s))\ ds$

Why does this make sense? If you take limits of both sides, and note that $y(t) = \lim_n y_{n+1} = \lim_n y_n$, then $y(t)$ is a solution to the integral equation, and hence a solution to the differential equation. The next question you should ask is under what hypotheses on f does this limit exist? It turns out that sufficient hypotheses are the f and f_y be continuous at $(0, 0)$. These are exactly the hypotheses given in your existence/uniqueness theorem 2.

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Note: If we stop this algorithm at a finite value of n, we expect $y_n(t)$ to be a very good approximate solution to the differential equation. This makes this method of iteration an extremely powerful tool for solving differential equations!

For a concrete example, I'll show you how to solve problem #3 from section $2-8$.

Use the method of picard iteration with an initial guess $y_0(t) = 0$ to solve:

$$y' = 2(y+1), \quad y(0) = 0.$$

Note that the initial condition is at the origin, so we just apply the iteration to this differential equation.

$$y_1(t) = \int_{s=0}^{t} f(s, y_0(s))\ ds = \int_{s=0}^{t} 2(y_0(s)+1)\ ds = \int_{s=0}^{t} 2\ ds = 2t.$$

Hence, we have the first guess is $y_1(t) = 2t$. Next, we iterate once more to get y_2:

$$y_2(t) = \int_{s=0}^{t} f(s, y_1(s))\ ds = \int_{s=0}^{t} 2(y_1(s)+1)\ ds = \int_{s=0}^{t} 2(2s+1)\ ds = \frac{2^2}{2!}t^2 + 2t.$$

Hence, we have the second guess $y_2(t) = \frac{2^2}{2!}t^2 + 2t$. Iterate again to get y_3:

$$y_3(t) = \int_{s=0}^{t} 2(y_2(s)+1)\ ds = \int_{s=0}^{t} 2\left(\frac{2^2}{2!}s^2 + 2s + 1\right)\ ds = \frac{(2t)^3}{3!} + \frac{(2t)^2}{2!} + 2t.$$

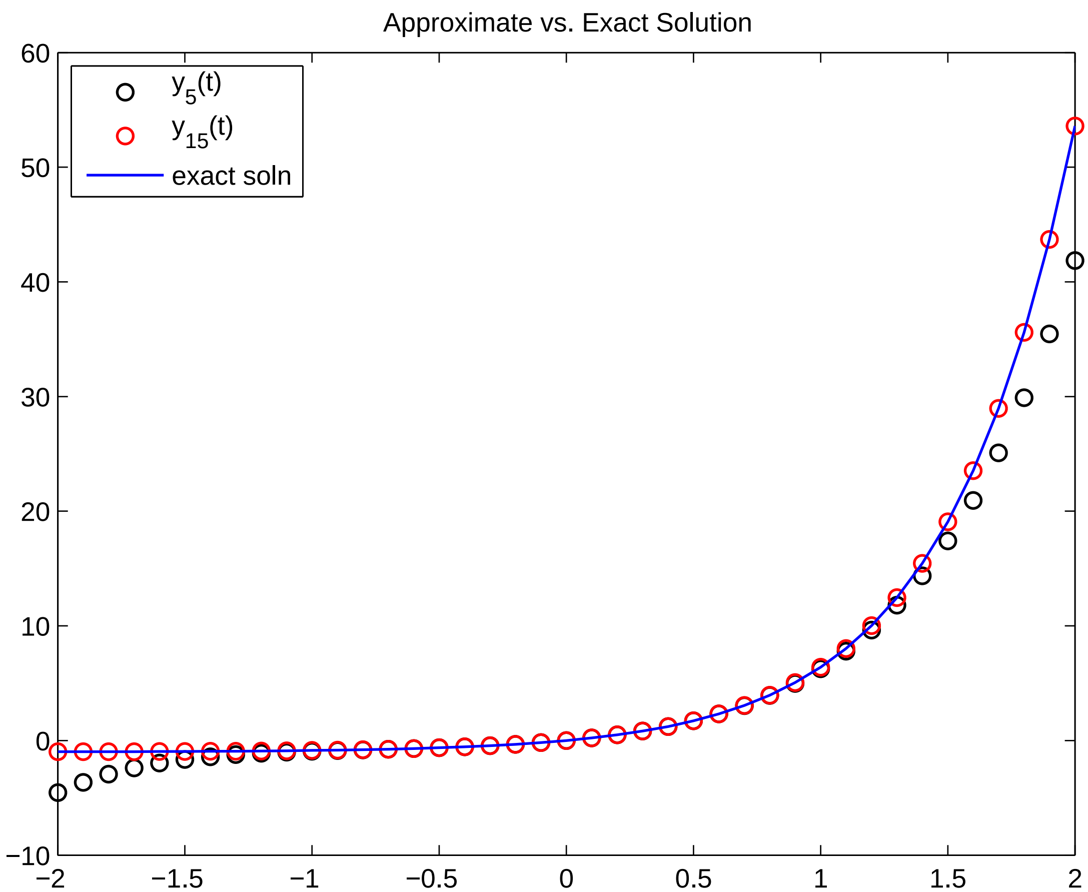
It looks like the pattern is

$$y_n(t) = \sum_{k=1}^{n} \frac{(2t)^k}{k!}$$

and hence the *exact* solution is given by

$$y(t) = \lim_{n\to\infty} y_n(t) = \sum_{k=1}^{\infty} \frac{(2t)^k}{k!} = \sum_{k=0}^{n} \frac{(2t)^k}{k!} - 1 = e^{2t} - 1.$$

If you plug this into the differential equation, you'll see we hit this one on the money. To demonstrate this solution actually works, below is a graph of $y_5(t)$, $y_{15}(t)$ and $y(t)$, the exact solution.


Approximate vs. Exact Solution
$y_5(t)$
$y_{15}(t)$
exact soln
60
50
40
30
20
10
0
−10
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2