EXACT DIFFERENTIAL EQUATIONS, A SECOND PERSPECTIVE

DAVID SEAL

One of the most difficult things when studying differential equations is knowing what type of differential equations that we can 'solve'. 'Solving' a differential can mean different things to different audiences, but in this course we focus on finding *analytic* solutions. I.e. finding a function $y = y(x)$ that solved the differential equation.

We pose the question 'What type of differential equations can we solve?' and we seek to classify a large category for which we can find a solution. Suppose we have a function $\psi(x, y) \equiv c$ that describes our unknown function $y = y(x)$. What differential equations could the function $y = y(x)$ satisfy? To see this, we take a differential of ψ:

$$0 = d(c) = d(\psi(x, y)) = \psi_x \; dx + \psi_y \; dy.$$

Dividing by dx (yeah, I know this isn't actually a 'number' but it works anyway) we have

$$0 = \psi_x + \psi_y \frac{dy}{dx}.$$

We now define functions $M(x, y) := \psi_x$ and $N(x, y) := \psi_y$, and hence our differential equation becomes

$$0 = M + N \frac{dy}{dx}. \tag{1}$$

If the function ψ is $C^2(\mathbb{R}^2)$, (i.e. it has two derivatives and they're both continuous - this is a very natural assumption to make), we want to enforce $\psi_{xy} = \psi_{yx}$. This amount to enforcing

$$M_y = (\psi_x)_y = \psi_{xy} = \psi_{yx} = (\psi_y)_x = N_x.$$

If this criteria holds, then we declare equation (1) to be an **exact differential equation**, and there exists a function $\psi(x, y) \equiv c$ which describes a solution to the differential equation.

I'll do problem #3 from section $2-6$ in your textbook. The differential equation is

$$(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0.$$

We first see if this is an exact differential equation.

$$M_y = \frac{\partial}{\partial y}(3x^2 - 2xy + 2) = -2x,$$

and

$$N_x = \frac{\partial}{\partial x}(6y^2 - x^2 + 3) = -2x.$$

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Since $M_y = N_x$, the differential equation is *exact* and hence there's a function $\psi(x, y) \equiv c$ that solves this differential equation. (Note: checking that $M_y = N_x$ is equivalent to checking that $\psi_{xy} = \psi_{yx}$.)

We have two equations we can integrate to find ψ:

$$\psi_x = M = 3x^2 - 2xy + 2$$
$$\psi_y = N = 6y^2 - x^2 + 3.$$

Integrating the first with respect to x, we have

$$\psi = \int (3x^2 - 2xy + 2)\ dx = x^3 - x^2 y + 2x + c(y).$$

The constant of integration may depend on y. To find the 'constant' we use the second equation:

$$\psi_y = \frac{\partial}{\partial y}(x^3 - x^2 y + 2x + c(y)) = -x^2 + c'(y) = 6y^2 - x^2 + 3.$$

This gives an ordinary differential equation to solve for c:

$$c'(y) = 6y^2 + 3.$$

Note that every x dropped out of the differential equation! If the original problem was not exact, then we would see x's here. Integrating once more yields

$$c(y) = 2y^3 + 3y + c_0,$$

and hence our solution is

$$\psi(x, y) = x^3 - x^2 y + 2x + 2y^3 + 3y + c_0 \equiv \text{const}.$$

We can group the constant c_0 onto the right hand side:

$$\psi(x, y) = x^3 - x^2 y + 2x + 2y^3 + 3y \equiv \text{const}.$$

This provides an *implicit* solution to the original differential equation.

The next thing to look at is what if the differential equation (1) is not an exact differential equation? Is it possible to turn it into an exact differential equation which we can solve? The answer is sometimes.

Read pages 98 and 99 for an example of how to use an integrating factor. Things are easier when it's possible to use integrating factors of the form $\mu = \mu(y)$ or $\mu = \mu(x)$ rather than $\mu = \mu(x, y)$.

1. Integrating Factor for Exact Differential Equations

We multiply

$$M\ dx + N\ dy = 0$$

by a yet to be determined integrating factor μ. The differential equation becomes

$$\mu M\ dx + \mu N\ dy = 0. \tag{2}$$

If we want equation (2) to be exact, we require

$$(\mu M)_y = (\mu N)_x.$$

Carrying out the product rule this gives

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x. \tag{3}$$

In principle this could be solved for μ, but this is now a partial differential equation which is harder to solve than the original one! Here we need to make a *reduction* *assumption* in order to make any progress. Here, we'll assume $\mu = \mu(y)$ and see what we can discover about μ. Equation (3) becomes

$$\mu' M + \mu M_y = \mu N_x$$

which becomes (after rearranging)

$$\frac{d}{dy}(\ln(\mu)) = \frac{\mu'}{\mu} = \frac{N_x - M_y}{M}.$$

If we inegrate both sides then exponentiate, this becomes

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\, dy\right), \tag{4}$$

under the assumption that $\mu = \mu(y)$**!** One can obtain a similar formula if we assume $\mu = \mu(x)$:

$$\mu(x) = \exp\left(\int -\frac{N_x - M_y}{N}\, dx\right). \tag{5}$$

How can we tell if this is going to work? (It doesn't always work). If we come up with integrands that we can integrate *and* after integrating we actually have an honest function of one of the variables, then the method will work.