RESEARCH STATEMENT

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1. Introduction

My research lies in geometry and to a lesser extent in geometric dynamics. My primary goal is to gain a better understanding of highly symmetric spaces by investigating the ways in which they can and cannot be deformed in larger classes of spaces. Rigidity theorems assert that genuine deformations are not possible. When rigidity fails, classifying the moduli of deformations is desirable.

I have explored diverse geometric settings with the above goal in mind. Prior investigations concerned higher rank lattice actions that exhibit hyperbolic behavior, cohomology of nonpositively curved locally symmetric spaces, equivariant packing problems in symplectic toric manifolds, aspects of geodesics in Riemannian spaces, spectral rigidity of homogeneous spaces, various curvature profiles for Riemannian spaces, and the asymptotic geometry of negatively curved geodesic metric spaces.

I summarize most of my work in the sections that follow. Sections 2, 3, and 4 are titled Rank-Rigidity, Constant Vector Curvature, and Almost-Isometries (quasi-isometries with multiplicative constant 1), respectively. These sections describe my more recent (in the last four years) work. Sections 5, 6, and 7 are titled Geodesics, Cohomology of Locally Symmetric Spaces, and Other, respectively. Some of the work in the Geodesics section was completed at MSU, but most was completed as a postdoc. Work described in the Cohomology of Symmetric Spaces section was completed as a graduate student and in the beginning of my postdoc years. Work in the Other section consists of a description of my thesis results, a brief description of a paper concerning symplectic toric packing that was written in graduate school, and a short section on homogeneous spaces that describes some work currently in progress.

2. Rank-Rigidity

Rank-rigidity in global Riemannian geometry is succinctly described by: “few spaces have the property that all geodesics infinitesimally appear to lie in constant curvature spaces – they ought to be classifiable”.

Rank-rigidity is well-established for manifolds with upper sectional curvature bounds. My recent work in this area primarily concerns rank-rigidity for manifolds with lower sectional curvature bounds.

A Riemannian $d$-manifold $M$ has extremal curvature $\epsilon \in \{-1, 0, 1\}$ if its sectional curvatures satisfy $\sec \leq \epsilon$ or $\sec \geq \epsilon$. For $M$ with extremal curvature $\epsilon$, the rank of a complete geodesic $\gamma : \mathbb{R} \to M$ is defined as the maximal number of linearly independent, orthogonal, and parallel vector fields $V(t)$ along $\gamma(t)$ satisfying $\sec(\dot{\gamma}, V)(t) \equiv \epsilon$. The manifold $M$ has (hyperbolic, Euclidean or spherical depending on whether $\epsilon$ is $-1, 0$ or $1$) rank at least $k$ if all its complete geodesics have rank at least $k$. 
Riemannian manifolds with $\sec \leq \epsilon$ and positive rank are known to be rigid. Finite volume Riemannian manifolds with bounded nonpositive sectional curvatures and positive Euclidean rank are locally reducible or locally isometric to symmetric spaces of nonpositive curvature [2, 15]. Generalizations include [21] and [86]. Closed Riemannian manifolds with $\sec \leq -1$ and positive hyperbolic rank are locally isometric to negatively curved symmetric spaces [37]; this fails in infinite volume [16]. Finally, closed Riemannian manifolds with $\sec \leq 1$ and positive spherical rank are locally isometric to positively curved, compact, rank one symmetric spaces [79].

Rank rigidity results are less definitive in the $\sec \geq \epsilon$ curvature settings. Hyperbolic rank rigidity results for manifolds with $-1 \leq \sec \leq 0$ first appeared in [18].

In recent joint work with J. Wolfson [74], we proved:

**Theorem 2.1.** A complete and finite volume three-manifold with $\sec \geq -1$ and positive hyperbolic rank is real hyperbolic.

To our knowledge, this is the first hyperbolic rank rigidity theorem without an upper curvature bound assumption. Theorem 2.1 is a special case of a more general classification for three-manifolds with *constant vector curvature* described below. It seems natural to pursue the following:

**Conjecture 2.1.** A complete and finite volume manifold with $\sec \geq -1$ and positive hyperbolic rank is locally symmetric.

In general, rank-rigidity fails in the $\sec \geq 0$ case [81, 39]. However, in dimension three, Euclidean rank rigidity holds in complete generality, as recently demonstrated in joint work with R. Bettiol [6]:

**Theorem 2.2.** Let $M$ be a complete three-manifold. Then $M$ has positive Euclidean rank if and only if the universal covering of $M$ is isometric to a product.

In the above theorem, $M$ having positive Euclidean rank in the absence of a curvature assumption means that every complete geodesic in $M$ admits a parallel vector field making curvature zero with the velocity field of the geodesic. No assumption is made on curvatures or on the volume of the manifold; it is the first such rank-rigidity theorem to our knowledge. We hope to generalize this theorem to slightly higher dimensions (e.g. four). In another direction, preliminary work with R. Bettiol suggests a positive answer to the following:

**Conjecture 2.2.** Assume that $M$ is a complete three-manifold with the property that for each complete geodesic $\gamma \subset M$ there exists a constant $k := k(\gamma)$ such that $\gamma$ admits a parallel vector field making curvature $k$ with the velocity field of $\gamma$. Then either $M$ is a space form or the universal covering of $M$ is isometric to a product.

As part of an ongoing collaboration with R. Shankar and R. Spatzier [72], we prove the first spherical rank-rigidity theorems for manifolds with $\sec \geq 1$. Before describing these results, it is instructive to make a comparison with the definitive spherical rank-rigidity result [79] for manifolds with $\sec \leq 1$.

In both cases, unit-speed geodesics $\gamma : \mathbb{R} \to M$ admit a Jacobi field $J(t) = \sin(t)V(t)$ where $V(t)$ is a normal parallel field along $\gamma$ contributing to its rank. Hence, for each $p \in M$, the tangent sphere of radius $\pi$ is contained in the singular set for $\exp_p : T_pM \to M$. In a symmetric space with $\frac{1}{4} \leq \sec \leq 1$, the first conjugate point along a unit-speed geodesic occurs at time $\pi$, the soonest time allowed by the curvature assumption $\sec \leq 1$. Consequently, the rank assumption is
an assumption about the locus of first singularities of exponential maps when sec \( \leq 1 \). In symmetric spaces with \( 1 \leq \text{sec} \leq 4 \), the first and second conjugate points along a unit-speed geodesic occur at times \( \pi/2 \) and \( \pi \), respectively. Therefore, when rank-rigidity holds in the sec \( \geq 1 \) setting, the rank assumption is an assumption about the locus of second singularities of exponential maps. Concerning first singularities of exponential maps, I proved the following theorem as a consequence of a new conjugate radius estimate in [67]:

**Theorem 2.3.** Assume that \( M \) is a simply-connected Riemannian manifold with sec \( \geq 1 \) in which the first conjugate point along each unit-speed geodesic occurs at time \( t = \frac{\pi}{2} \). Then \( M \) is isometric to a compact rank one symmetric space.

Recovering information about the first singularities of exponential maps from data about the location of later singularities is a serious difficulty in the following:

**Conjecture 2.3.** A Riemannian manifold with sec \( \geq 1 \) and positive spherical rank is locally isometric to a compact rank one symmetric space.

We can only verify this conjecture (after four years of collaboration) in some special cases. Note that amongst the compact rank one symmetric spaces of dimension \( d \), normalized so that sec \( \geq 1 \), only the curvature one spheres and complex projective spaces with constant holomorphic curvatures four have spherical rank at least \( d - 2 \).

**Theorem 2.4.** An odd dimensional Riemannian \( d \)-manifold with \( d \geq 3 \), sec \( \geq 1 \), and spherical rank at least \( d - 2 \) has constant sectional curvatures sec \( \equiv 1 \). A \( d \)-sphere with \( d \geq 3 \), \( d \neq 6 \), sec \( \geq 1 \) and spherical rank at least \( d - 2 \) has constant sectional curvatures sec \( \equiv 1 \).

**Theorem 2.5.** Let \( M \) be an even dimensional Riemannian \( d \)-manifold with \( d \geq 4 \), sec \( \geq 1 \), and spherical rank at least \( d - 2 \). If \( M \) does not have constant sectional curvatures i.e., sec \( \neq 1 \), then \( M \) satisfies:

1. Every vector \( v \in SM \) is contained in a 2-plane section \( \sigma \) with sec(\( \sigma \)) > 1.
2. The geodesic flow \( \phi_t : SM \to SM \) is periodic with \( 2\pi \) a period.
3. There exists an almost complex structure \( J : TM \to TM \) if \( M \) is simply connected.
4. If \( M \) is simply connected and if sec < 9, then every geodesic in \( M \) is simple, closed, and of length \( \pi \). Moreover, \( M \) is homotopy equivalent to \( \mathbb{CP}^{d/2} \).

**Theorem 2.6.** A Kählerian manifold with sec \( \geq 1 \), real dimension \( d \geq 4 \), and spherical rank at least \( d - 2 \) is isometric to a symmetric \( \mathbb{CP}^{d/2} \) with holomorphic curvatures equal to 4.

Our method of proving Theorem 2.6 yields a new Schur type theorem for Kählerian manifolds (see the preliminary report [73]). Let \( M \) be Riemannian, \( p \in M \), and \( v \in S_pM \). The Jacobi operator \( J_v : v^+ \to v^+ \) is the self-adjoint linear map defined by \( J_v(w) = R(v,v)w \), where \( R \) denotes the curvature tensor. Its eigenvalues determine the sectional curvatures of planes containing \( v \). Given an integer \( 0 \leq k < \text{dim}(M)/2 \), we say that a point \( p \in M \) has coisotropy rank at most \( k \) if there exists a constant \( \kappa(p) \in \mathbb{R} \) such that rank(\( J_v - \kappa(p) \text{Id} \)) \( \leq k \) for each \( v \in S_pM \); when all \( p \in M \) satisfy this property, we say that \( M \) has coisotropy rank at most \( k \).
**Theorem 2.7.** Let $M$ be a complete, simply-connected Kählerian manifold of real dimension $2n \geq 4$. If $M$ has coisotropy rank at most one, then $\kappa : M \to \mathbb{R}$ is constant. Moreover,

1. if $\kappa > 0$, then $M$ is isometric to a symmetric metric on $\mathbb{CP}^n$ having constant holomorphic curvatures $4\kappa$.
2. if $\kappa < 0$, then $M$ is isometric to a symmetric metric on $\mathbb{CH}^n$ having constant holomorphic curvatures $4\kappa$.
3. if $\kappa = 0$, then the open set of non-isotropic points $O$ admits an $(n-1)$-dimensional parallel distribution that is tangent to a foliation by complete and totally geodesic leaves isometric to $\mathbb{C}^{n-1}$.

3. **Constant Vector Curvature**

A complete Riemannian manifold with extremal curvature $\epsilon$ ($\sec \geq \epsilon$ or $\sec \leq \epsilon$) has constant vector curvature $\epsilon$ if every tangent vector $v$ to $M$ is contained in a tangent plane $\sigma$ with $\sec(\sigma) = \epsilon$.

This curvature condition was introduced in joint work with J. Wolfson [74] as an attempt to introduce a pointwise notion of geometric rank. In dimension three, the eight Thurston geometries have constant vector curvature. We originally thought that the constant vector curvature condition would furnish a characterization of the Thurston geometries. This turned out not to be the case. Theorem 2.1 above is easily deduced from the following rigidity result:

**Theorem 3.1.** Suppose that $M$ is a finite volume three-manifold with constant vector curvature. If $\sec \leq -1$, then $M$ is real hyperbolic. If $\sec \geq -1$ and $M$ is not real hyperbolic, then its universal covering is isometric to a left-invariant metric on one of the Lie groups $\text{Sol}$ or $\widetilde{\text{SL}}(2, \mathbb{R})$ with sectional curvatures having range $[-1, 1]$.

The left-invariant metrics on $\widetilde{\text{SL}}(2, \mathbb{R})$ belong to a one parameter family of pairwise nonisometric metrics with the same curvature tensor [50]. More precisely, let $R_0$ denote an algebraic curvature tensor on Euclidean space $\mathbb{E}^n$. A Riemannian $n$-manifold $M$ is curvature homogeneous with model $R_0$ if for each $p \in M$ there is a linear isometry $L : T_pM \to \mathbb{E}^n$ with $L^*(R_0) = R_p$ where $R_p$ denotes the curvature tensor of $M$ at the point $p$.

The assumption that $M$ has finite volume is necessary in Theorem 3.1 as we demonstrated in another joint work with J. Wolfson [75]. In the following theorem, the metrics corresponding to constant functions coincide with those in Theorem 3.1.

**Theorem 3.2.** There is a construction that associates to each positive smooth function $F : S^1 \to \mathbb{R}$ a complete and curvature homogeneous Riemannian metric $g_F$ on $\text{SL}(2, \mathbb{R})$. In this construction, the following are equivalent:

1. $F$ is constant.
2. The metric $g_F$ is left-invariant.
3. $(\text{SL}(2, \mathbb{R}), g_F)$ Riemannian covers a finite volume manifold.

The metrics in Theorem 3.2 are all modeled on the same algebraic curvature tensor. We also prove that the moduli of these metrics is infinite dimensional:
Theorem 3.3. Let $F$ and $G$ be positive smooth functions on the circle. Then there exists a diffeomorphism $\Phi: SL(2, \mathbb{R}) \to SL(2, \mathbb{R})$ such that $\Phi^*(g_G) = g_F$ if and only if there exists a diffeomorphism $\phi: S^1 \to S^1$ such that $F = \phi^*(G)$.

Theorem 3.3 yields the first infinite dimensional moduli space of curvature homogeneous spaces with curvature tensor modeled on an irreducible homogeneous spaces. Earlier examples based on reducible homogeneous spaces (e.g. $\mathbb{H}^2 \times \mathbb{R}$) are described in [78, 8]. It is worth mentioning the following conjecture that is attributed to Gromov by Berger in [4].

Conjecture 3.1. If $M$ is compact and $R_0$ is an algebraic curvature tensor, then the moduli space of metrics on $M$ with curvature model $R_0$ is finite dimensional.

Preliminary investigations with J. Wolfson suggest that infinite dimensional moduli can occur on the three-sphere. Returning to the structure of constant vector curvature three-manifolds, we have the following result also from [74].

Theorem 3.4. Assume that $M^3$ has constant vector curvature and extremal curvature zero. If $M$ has finite volume, then the open subset of nonisotropic points in $M$ admits a local product structure.

The assumption that $M$ has finite volume is necessary in Theorem 3.4 as illustrated by examples of Sekigawa [78] consisting of real analytic Riemannian metrics on $\mathbb{R}^3$ that are curvature homogeneous with model $\mathbb{H}^2 \times \mathbb{R}$, but nowhere locally isometric to a product. Moreover, the local product structure in Theorem 3.4 need not arise from the universal covering having a global product structure. This fact is illustrated by Gromov’s graph manifolds, manifolds obtained by gluing constructions (see e.g. [6]). Conjecturally, three manifolds admitting a constant vector curvature zero metric are graph manifolds. However, there are no known obstructions to admitting a constant vector curvature zero metric, while there are for graph manifolds.

Problem 1. Find a closed three manifold that does not admit a Riemannian metric with constant vector curvature zero and with pointwise signed curvatures ($\sec_p \leq 0$ or $\sec_p \geq 0$ for each $p \in M$).

Theorems 3.1 and 3.4 illustrate some rigidity properties of the constant vector curvature condition. The proofs involved a dynamical study of the evolution of Christoffel symbols along a vector field that lies in the intersection of all curvature $\epsilon$ planes. In contrast, the class of constant vector curvature and extremal curvature one three-manifolds is quite flexible.

Locally homogeneous examples include Berger’s metrics on three-spheres (rescaled to have constant vector curvature one), quotients of the Heisenberg group, and quotients of a one parameter family of metrics on $T_1 \mathbb{H}^2$ obtained by a scaling of the Sasaki metric analogous to that of the Berger spheres. The compact locally homogeneous examples are all circle bundles over a surface. In work in preparation with J. Wolfson, we are employing PDE methods to produce an infinite dimensional moduli of perturbations of the above locally homogeneous examples.

Theorem 3.5. Let $(M, g)$ be a compact locally homogeneous constant vector curvature one three manifold with scalar curvature $S = 4 - 2(2c + 1)$ for some constant $c \neq 1$. 
There is a deformation procedure that associates to each smooth basic (constant on circle fibers) function $\xi : M \to \mathbb{R}$ satisfying
$$\int_M \xi \, dvol_g = \int_M 2(1 - c) \, dvol_g$$
a Riemannian metric $\bar{g} := \bar{g}(c, \xi)$ on $M$ with constant vector and extremal curvature one and with scalar curvature $S = 4 + 2\lambda$ where $\lambda = e^{2h} - 3$ and $h : M \to \mathbb{R}$ satisfies the elliptic equation
$$\Delta_g h = \xi - 2(1 - c).$$

The Riemann manifold in Theorem 3.5 Riemannian submerse to a metric on the surface base with scalar curvature $S + 3$. Combining Theorem 3.5 with the classical results of Kazdan-Warner [44] allow one to prescribe the scalar curvature of a constant vector curvature and extremal curvature one metric on a compact three manifold so long as the base scalar curvature function $S + 3$ is not obstructed by Gauss-Bonnett.

4. Almost-Isometries

Quasi-isometries are the principal morphisms in asymptotic geometry, and in particular, in geometric group theory. Given constants $K \geq 1$ and $C > 0$, a $(K, C)$-quasi-isometry between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is a map $f : X \to Y$ satisfying $\frac{1}{K} d_X(a, b) - C \leq d_Y(f(a), f(b)) \leq K d_X(a, b) + C$ and satisfying $N_C(f(X)) = Y$ where $N_C(\cdot)$ denotes the metric $C$-neighborhood in $Y$. Note that $(K, 0)$ maps are bilipschitz maps. We define almost-isometries as $(1, C)$ quasi-isometries.

If $M$ is a closed manifold and $g_1$ and $g_2$ are two Riemannian metrics on $M$, then letting $d_1$ and $d_2$ denote the two $\pi_1(M)$-invariant path metrics on $X = \tilde{M}$ induced by the lifted Riemannian metrics $\tilde{g}_1$ and $\tilde{g}_2$, then there exists a $\pi_1(M)$-equivariant quasi-isometry $f : (X, d_1) \to (X, d_2)$.

In recent joint work with J.-F. Lafont and W. van Limbeek [49], we exhibit examples of almost-isometries between universal coverings of negatively curved surfaces that cannot be realized equivariantly.

**Theorem 4.1.** Let $M$ be a closed surface of genus $\geq 2$, and $k \geq 1$ an integer. One can find a $k$-dimensional family $F_k$ of Riemannian metrics on $M$, all of curvature $\leq -1$, with the following property. If $g, h$ are any two distinct metrics in $F_k$, then
- $\text{Area}(M, g) = \text{Area}(M, h)$.
- the lifted metrics $\tilde{g}, \tilde{h}$ on the universal cover $X$ are almost-isometric.
- the lifted metrics $\tilde{g}, \tilde{h}$ on the universal cover $X$ are not isometric.

In the above Theorem 4.1, we think of the almost isometries between the lifted metrics on $X$ as being exotic since they cannot be realized equivariantly with respect to the two natural $\pi_1(M)$-actions on $X$. Indeed, the existence of a $\pi_1(M)$-equivariant almost-isometry between the two lifted metrics on $X$ implies that the two metrics on $M$ have equal marked length spectra (see [43], for example), and are therefore isometric by [20, 59].

As another example of a theorem asserting that almost-isometries exist in abundance, we also have the following:

**Theorem 4.2.** Let $(M_1, d_1), (M_2, d_2)$ be any pair of closed surfaces equipped with locally $\text{CAT}(-1)$ metrics, and let $(X_i, \tilde{d}_i)$ be their universal covers. Then one can
find real numbers $0 < \lambda_i \leq 1$, with $\max\{\lambda_1, \lambda_2\} = 1$, having the property that 
$(X_1, \lambda_1 d_1)$ is almost-isometric to $(X_2, \lambda_2 d_2)$.

While Theorems 4.1 and 4.2 assert the existence of almost isometries, the proofs are not at all constructive; the existence of these almost isometries is still quite puzzling. The proof proceeds indirectly by first establishing that boundaries of such spaces, when equipped with their visual metrics, are quasi-circles in the sense of Falconer and Marsh and hence classified up to bilipschitz equivalence by their Hausdorff dimension [22]. Results due to Bonk and Schramm [10] then guarantee the existence of an almost isometry. As the work of Falconer and Marsh only applies to boundary circles, we do not currently have an approach to the following:

**Conjecture 4.1.** Let $M$ be a closed manifold of dimension $\geq 3$ supporting negatively curved Riemannian metrics. Then there exists a pair of equal volume Riemannian metrics $g_1$ and $g_2$ with $\sec \leq -1$, and having the property that the universal covers $(\tilde{M}, \tilde{g}_1)$ are almost-isometric, but not isometric.

Our methods also exhibit a gap phenomenon for quasi-isometries:

**Theorem 4.3.** Let $(M_1, d_1), (M_2, d_2)$ be a pair of closed surfaces equipped with locally CAT(-1) metrics, and assume that their universal covers $(\tilde{X}_i, \tilde{d}_i)$ are not almost-isometric. Then there is a constant $\epsilon > 0$ with the property that any $(K, C)$-quasiiisometry from $(X_1, d_1)$ to $(X_2, d_2)$ must satisfy $K \geq 1 + \epsilon$.

To contrast the above mentioned results, joint work with A. Kar and J.-F. Lafont [43] demonstrates the rigidity of locally symmetric spaces from the point of view of almost-isometries:

**Theorem 4.4.** Let $(M, g_0)$ be a closed locally symmetric space modeled on quaternionic hyperbolic space, or on the Cayley hyperbolic plane, and let $g_1$ be a negatively curved Riemannian metric on $M$. Then $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are almost-isometric if and only if $(M, g_0)$ and $(M, g_1)$ are isometric.

**Theorem 4.5.** Let $M$ be a closed $n$-manifold equipped with Riemannian metrics $g_0$ and $g_1$ for which the universal coverings $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are almost-isometric. Further assume that the metrics satisfy any of the following conditions:

1. $n = 2$, $g_0$ is a flat metric, and $g_1$ is arbitrary, or
2. $n = 2$, $g_0$ is a real hyperbolic metric, and $g_1$ satisfies $\Vol(g_0) \geq \Vol(g_1)$, or
3. $n \geq 3$, $g_0$ is a negatively curved locally symmetric metric, and $g_1$ satisfies $\Vol(g_0) \geq \Vol(g_1)$, or
4. $n \geq 5$, $g_0$ is an irreducible, higher rank, nonpositively curved locally symmetric metric, and $g_1$ is conformal to $g_0$ and satisfies $\Vol(g_0) \geq \Vol(g_1)$,
5. $n \geq 6$, $g_0$ is a locally symmetric metric modeled on a product of negatively curved symmetric spaces of dimension $\geq 3$ (suitably normalized), and $g_1$ is any metric satisfying $\Vol(g_0) \geq \Vol(g_1)$.

Then the universal covers $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are isometric. In particular, $(M, g_0)$ is isometric to $(M, g_1)$ by Mostow rigidity in cases (3) - (5).

5. Geodesics

This section summarizes my work on geodesics ([7, 46, 69, 68, 70]. Throughout, $M$ denotes a complete Riemannian manifold.
Blocking Light. Let \( p, q \in M \) and let \( \Gamma(p, q) \) denote the set of geodesic segments in \( M \) joining \( p \) to \( q \). A set \( X \subset M \setminus \{p, q\} \) blocks \( \Gamma(p, q) \) if the interior of every geodesic in \( \Gamma(p, q) \) intersects \( X \). The blocking number \( b(p, q) \in \mathbb{N} \cup \{\infty\} \) is defined as the fewest number of points needed to block \( \Gamma(p, q) \). When \( b : M \times M \to \mathbb{N} \cup \{\infty\} \) is finite (bounded), \( M \) is said to have (uniform) finite blocking.

Blocking light originated in billiard systems ([24, 32, 38, 56, 57]). Blocking in Riemannian spaces has also received considerable attention (c.f. [14, 26, 27, 33, 35, 40, 46, 69, 84]). A simple yet elegant argument proves the following (see e.g. [35]):

**Theorem:** Each closed flat \( n \)-dimensional manifold has uniform finite blocking with a constant depending only on the dimension \( n \).

In joint work with Lafont, we adapted the proof of the above theorem in [46] to prove that compact quotients of Euclidean buildings have uniform finite blocking. In the same paper, we made the following conjecture, that was also made independently by Burns and Gutkin in [14]:

**Conjecture 5.1.** A closed Riemannian manifold \( M \) has finite blocking if and only if \( M \) is flat.

We provided the following evidence for this conjecture ([14, 46]):

**Theorem 5.1.** If \( M \) is closed, nonpositively curved, and has finite blocking, then \( M \) is flat.

Conjecture 5.1 is supported by a few more results: The fundamental group is virtually nilpotent for a closed manifold with uniform finite blocking [14]. A closed surface with finite blocking and not covered by \( S^2 \) is flat as proved by Bangert and Gutkin [3].

An approach to Conjecture 5.1 is to prove: 1) finite blocking implies uniform finite blocking, and 2) (uniform) finite blocking implies no conjugate points. Indeed, if 1) holds, then \( \pi_1(M) \) is virtually nilpotent by [14]. By work of Lebedeva [51], generalizing the Burago and Ivanov [13] resolution of the Hopf conjecture, 2) then implies that \( M \) is flat. Lafont and I used this approach to prove that closed manifolds with regular finite blocking, a continuity condition on blocking sets, are necessarily flat in [46].

Compact rank one symmetric spaces have the following CROSS blocking property. If \( 0 < d(x, y) < \text{diam}(M) \), then \( b(x, y) \leq 2 \). Spheres additionally satisfy sphere blocking, \( b(x, x) = 1 \) for every \( x \in M \).

In joint work with J. Souto [69] we proved the following:

**Theorem 5.2.** A closed Riemannian manifold \( M \) has Cross and sphere blocking if and only if \( M \) is isometric to a round sphere.

**Generalizations of the 3 Gap Theorem.** The classical Three Gap Theorem [80, 83] asserts that for \( n \in \mathbb{N} \) and \( p \in \mathbb{R} \), there are at most three distinct distances between consecutive elements in the subset of \([0, 1)\) consisting of the reductions modulo 1 of the first \( n \) multiples of \( p \). Regarding this as a statement about (isometric) rotations of the circle, I found Riemannian generalizations jointly with I Biringer [7].

Let \( M \) denote a complete Riemannian manifold with distance function \( d \) and let \( X \subset M \) be a finite subset. For \( x \in X \), denote the distance from \( x \) to its nearest neighbor in \( X \) by \( \text{nnd}(x, X) = \min_{y \in X \setminus \{x\}} d(x, y) \). Let \( \text{NND}(X) = \)
\{\text{nnd}(x, X) \mid x \in X\} denote the set of all nearest neighbor distances in \(X\). Our first generalization of the 3-gap theorem concerns isometries.

**Theorem 5.3.** For each \(k \in \mathbb{N}, \kappa \in \mathbb{R}, \) and \(D > 0\), there is a constant \(K(k, \kappa, D) \in \mathbb{N}\) such that for any complete Riemannian \(k\)-manifold with \(\sec \geq \kappa\) and \(\text{diam}(M) \leq D\), and for any isometry \(I, p \in M, \) and \(n \in \mathbb{N}\),

\[|\text{NND}([I^i(p) \mid i = 0, \ldots, n])| \leq K.\]

When \(\kappa = 0\), we can take \(K = 3^k + 1\), though this is likely far from sharp. In a different direction, we thought of the traditional 3-gap theorem as a statement about the distribution of points along a geodesic in the circle, leading to the following definition.

We say that \(M\) has **bounded geodesic combinatorics** if there exists \(K \in \mathbb{N}\) such that for every unit-speed geodesic \(\gamma : \mathbb{R} \to M, T \in \mathbb{R}\) and \(n \in \mathbb{N}\),

\[|\text{NND}([\gamma(iT) \mid i = 0, \ldots, n])| \leq K.\]

We completely classify Riemannian surfaces with this property.

**Theorem 5.4.** A closed Riemannian surface has bounded geodesic combinatorics if and only if it is isometric to a flat torus, a round metric on the projective plane, or a metric on the two sphere with all geodesics simple, closed, and of equal length.

The argument is quite two-dimensional. When \(M\) is not covered by the 2-sphere, we used classical results concerning minimizing geodesics \([25]\) to prove that for each free homotopy class \(\Gamma\) of closed curves, \(M\) admits a foliation by closed geodesics in \(\Gamma\). A theorem due to Innami \([41]\) implies that \(M\) is flat. Generalizing Innami’s theorem to higher dimensions is a challenge.

**Conjecture 5.2.** Assume that \(M\) is a closed aspherical Riemannian manifold with the property that for each free homotopy class \(\Gamma\) of closed curves in \(M, M\) is foliated by closed geodesics in \(\Gamma\). Then \(M\) is flat.

When \(M\) is covered by the two-sphere, we work quite hard to prove that \(M\) has all geodesics closed. In the case when \(M\) is the two sphere, all geodesics are simple, closed, and of equal length by a result due to Gromoll and Grove \([29]\). In the case when \(M\) is the projective plane, a result due to Pries \([62]\) implies the metric is round. The method of Pries is very two-dimensional and dynamical in nature. Preliminary results with my Ph.D. student S. Lin support the following:

**Conjecture 5.3.** A Riemannian metric on \(\mathbb{RP}^n\) with all geodesics closed has constant positive sectional curvatures.

**Spherical Points.** Götz and Rybarksi asked whether round spheres are the only convex surfaces with the property that every pair of points is joined either by a unique minimizing geodesic or by infinitely many minimizing geodesics \([28, 19]\). Zamfirescu answered this question affirmatively for \(C^3\)-smooth convex surfaces \([88]\). In the short and easy note \([68]\), I similarly characterized round spheres amongst all smooth Riemannian surfaces.

A point \(p\) in a closed Riemannian manifold is defined to be **weakly spherical** if for each distinct point \(q \in M\), the set of minimizing geodesics joining \(p\) to \(q\) has either one element or at least three elements. A point \(p \in M\) is defined to be **strongly spherical** if for each distinct point \(q \in M\), the set of minimizing geodesics joining \(p\) to \(q\) has either one element or infinitely many elements.
Theorem 5.5. If $M$ is a closed Riemannian surface with a weakly spherical point realizing the injectivity radius, then $M$ is isometric to a constant curvature sphere.

Theorem 5.6. If $M$ is closed Riemannian manifold with all points weakly spherical, then $M$ is simply connected.

Length Spectrum. The short note [70], written jointly with C. Sutton, concerns the length spectrum of a Riemannian manifold. The length spectrum of a Riemannian manifold consists of the set of lengths of all closed geodesics in $M$. We could not find examples in the literature of Riemannian manifolds with indiscrete length spectrum and so proved:

Proposition 5.7. Let $M$ be a smooth manifold of dimension at least three and $\gamma : S^1 \to M$ a smooth simple closed curve. Then there exists a smooth Riemannian metric $g$ on $M$ with respect to which $\gamma$ is a closed geodesic and its length is an accumulation point of the length spectrum of $g$.

We also give an example of a Riemannian metric with an uncountable length spectrum. These constructions are necessarily only smooth since we also prove via Morse theory:

Theorem 5.8. If $(M, g)$ is a closed real analytic Riemannian manifold, then its length spectrum forms a discrete subset of the real line.

6. Cohomology of Locally Symmetric Spaces

This section describes joint work with J.-F. Lafont on the (bounded) cohomology of locally symmetric spaces of noncompact type.

One of the central topological questions about closed locally symmetric $M^n$ was positivity of their simplicial volumes, as conjectured in Gromov’s famous paper [31], and proved therein for the hyperbolic spaces (see also [11, 12, 42, 65, 85]). In [47], we proved the following:

Theorem 6.1. Closed locally symmetric spaces of noncompact type have positive simplicial volume.

An argument due to Thurston [85], reduces proving positivity of the simplicial volume to the more geometric problem of finding an appropriate procedure for straightening singular simplices. Our main contribution was to introduce a barycentric straightening of simplices in symmetric spaces of noncompact type, building on the earlier work Besson-Courtois-Gallot and Connell-Farb [5, 17]. Still open is the more general conjecture due to Gromov (cf [52]):

Conjecture 6.1. Closed Riemannian manifolds with nonpositive curvature and negative Ricci curvature have positive simplicial volume.

Conjecture 6.1 is easily reduced to the case of a closed geometric rank one manifold of nonpositive curvature. The major challenge lies in defining a straightening operator on simplices in such a way that top dimensional simplices have uniformly bounded volumes.

The simplicial volume is defined by equipping the simplicial chain complex of $M^n$ with the $l^1$-norm on chains. A dual theory is obtained by equipping the real cochain complex of $M^n$ with the $l^\infty$-norm, leading to the bounded cohomology groups $\hat{H}^k(M^n, \mathbb{R})$ ([31]). The inclusion of the bounded cochain complex into the
ordinary cochain complex defines a comparison homomorphism \( c : \hat{H}^k(M^n, \mathbb{R}) \to H^k(M^n, \mathbb{R}) \). Few examples are known of cohomology classes in the image of this homomorphism. The dual formulation of Theorem 6.1 states that the comparison homomorphism is surjective when \( k = n \). This leads to the following problem:

**Problem 2.** Provide new examples of cohomology classes in higher rank locally symmetric spaces in the image of the comparison homomorphism.

One approach to Problem 2 is to find examples of totally geodesically embedded higher rank locally symmetric spaces \( Y^m = \Gamma' \backslash G'K' \) inside higher rank locally symmetric spaces \( X^n = \Gamma \backslash G/K \) where the (included) fundamental class \( [Y^m] \in H_k(X^n, \mathbb{R}) \) has positive \( l^1 \)-norm. We view this approach as having two steps. First is to understand when \( [Y^m] \) is non-trivial in \( H_k(X^n, \mathbb{R}) \). Second is to detect when a homologically essential submanifold \( Y^m \) with positive simplicial volume retains a positive \( l^1 \)-norm in \( M^n \). We have addressed the first step in [48], giving a criterion in terms of the naturally induced embedding \( Y^k_u = \Gamma'_u \backslash K' \hookrightarrow \Gamma_u \backslash K = X^m_u \) between the associated nonnegatively curved dual compact type symmetric spaces.

**Theorem 6.2.** Let \( Y^m \hookrightarrow X^n \) be a compact totally geodesic submanifold of noncompact type inside a compact locally symmetric space of noncompact type, and denote by \( \rho \) the map on cohomology \( H^m(X_u; \mathbb{R}) \to H^m(Y_u; \mathbb{R}) \simeq \mathbb{R} \) induced by the embedding \( Y_u \hookrightarrow X_u \). Then we have:

- if \( [Y^m] = 0 \in H_m(X^n; \mathbb{R}) \) then the map \( \rho \) is identically zero.
- if \( \rho \) is identically zero, and \( m \leq m(\mathfrak{g}) \), where \( m(\mathfrak{g}) \) is the Matsushima constant corresponding to the Lie algebra \( \mathfrak{g} \) of the Lie group \( G \), then we have that \( [Y^m] = 0 \in H_m(X^n; \mathbb{R}) \).

Our proof is an adaptation of an argument of Matsushima [55] and relies on the existence of certain compatible maps (the Matsushima maps) from the real cohomology of the pair of nonnegatively curved duals \( (X_u, Y_u) \) to the real cohomology of the nonpositively curved pair \( (X^n, Y^m) \). It is reasonable to ask whether this map can be realized geometrically. Extending work of Okun [58], we show that this can sometimes be achieved rationally.

**Theorem 6.3.** Assume that \( Y^m \hookrightarrow X^n \) is a totally geodesic embedding of compact, locally symmetric spaces of noncompact type. Furthermore, assume that the map \( G'_u \curvearrowleft G_u \) induced by the the inclusion \( Y \hookrightarrow X \) is a \( \pi_i \)-isomorphism for \( i < m \) and a surjection on \( \pi_m \). Then there exists a finite cover \( \tilde{X} \) of \( X^n \), and a connected lift \( \tilde{Y} \subset \tilde{X} \) of \( Y^m \), with the property that there exists a tangential map of pairs \( (\tilde{X}, \tilde{Y}) \to (X_u, Y_u) \). If in addition we have \( rk(G_u) = rk(K) \) and \( rk(G'_u) = rk(K') \), then the respective tangential maps induce the Matsushima maps on cohomology.

While the hypotheses of this last theorem are fairly technical, we show that there are examples of inclusions \( Y^m \hookrightarrow X^n \) satisfying the hypotheses. Though our interest was initially in finding bounded cohomology classes, the Okun map may have applications to the more classical Steenrod realization problem.

**Problem 3.** Give an example of a closed locally symmetric space \( M \) and a homology class in \( M \) that cannot be represented as the image of a closed manifold.

In [9], Bohr, Hanke, and Kotschick give an example of a degree seven homology class in \( Sp(2, \mathbb{H}) \) that is not representable as the image of a closed manifold. J. Lafont and I have an approach to Problem 3 that involves using the example from [9].
and the Okun map in order to give an example in a compact locally symmetric space of noncompact type dual to $Sp(2, \mathbb{H})$. The argument for non-representability in [9] utilizes cohomology with $\mathbb{Z}_3$ coefficients. While the Matsushima map is between real cohomologies, we hope that the geometric Okun map will not lose the required information on torsion.

7. Other: Weakly Hyperbolic Actions, Packing Symplectic Toric Manifolds, Problems about Homogeneous Spaces

Weakly Hyperbolic Actions. My doctoral thesis was concerned with the study of large group actions [45], and in particular, the Zimmer program of classifying ergodic volume preserving actions of lattice subgroups of noncompact higher real rank simple Lie groups on compact manifolds [89].

Roughly twenty years before the beginning of Zimmer’s classification program, Smale posed the problem of classifying Anosov diffeomorphisms on closed manifolds. The work in [23] and [53] classifies Anosov diffeomorphisms on infranilmanifolds (a generalization of tori). The notion of weakly hyperbolic group actions was introduced by Margulis and Qian [54] in work related to Zimmer’s program. Weak hyperbolicity of an action is an analogue of the Anosov property for a diffeomorphism. Indeed, weakly hyperbolic actions of the integers are generated by an Anosov diffeomorphism. In [66] I proved the following:

**Theorem 7.1.** Let $\rho : \Gamma \to Diff^2(M^n)$ be a volume preserving weakly hyperbolic action by $C^2$ diffeomorphisms. Then the action $\rho$ is ergodic.

When $\Gamma = \mathbb{Z}$, Theorem 7.1 specializes to Anosov’s classical result [1] that volume preserving $C^2$ Anosov diffeomorphisms are ergodic. However, to account for a general group action as opposed to the dynamics of a single diffeomorphism, the proof proceeds differently than the existing proofs for ergodicity. It is based on microlocal regularity results in [63] relating Sobolev classes of functions measured tangentially with respect to leaves of an absolutely continuous (satisfying a Fubini type theorem) foliation to the ordinary Sobolev classes of functions. The analysis involved also plays an important role in the proof of Theorem 7.3 below. Theorem 7.3 combined with a complementary theorem in [54] yields:

**Theorem 7.2.** Suppose $\Gamma$ is a higher rank lattice and let $\rho : \Gamma \to Diff^2(T^n)$ be a weakly hyperbolic volume preserving action covered by an action of $\Gamma$ on $\mathbb{R}^n$. Then (after passing to a finite index subgroup) $\rho$ is continuously semiconjugated to the affine action on $T^n$ coming from the induced action on first homology.

The Franks and Manning classification of Anosov diffeomorphism on tori asserts that an Anosov diffeomorphism $f$ on $T^n$ is equivalent (conjugate to by a homeomorphism) to the hyperbolic automorphism of the torus descending from the automorphism $f_* : H_1(T^n, \mathbb{Z}) \to H_1(T^n, \mathbb{Z})$. Theorem 7.2 provides strong evidence that an analogous classification holds for weakly hyperbolic actions of higher rank lattices on tori. The proof of Theorem 7.2 makes use of the rigidity properties of higher rank lattices in essential ways. In particular, it uses Zimmer’s cocycle superrigidity theorem and Kazhdan’s property (T) for these groups. It also relies technically on my following [66]:
Theorem 7.3. Let \( \rho : \Gamma \to \text{Diff}^2(T^n) \) be a weakly hyperbolic volume preserving action of a property (T) group covered by an action of \( \Gamma \) on \( \mathbb{R}^n \). Then the representation coming from the induced action of \( \Gamma \) on the first homology of \( T^n \) does not split non-trivially as a direct sum of subrepresentations, one of which is isometric.

Packing Symplectic Toric Manifolds. This work was joint with A. Pelayo. He initiated the study of ball packings in symplectic manifolds in which the balls being packed are equivariant with respect to an effective and Hamiltonian action of a torus. When the acting torus is half the dimension of the symplectic manifold, Pelayo classified exactly those manifolds admitting a density one packing. More precisely, he proved the following theorem in [60]:

Theorem 7.4. A symplectic-toric manifold \( (M^{2n}, \sigma, \psi) \) admits a full toric ball packing if and only if there exists \( \lambda > 0 \) such that

- if \( n = 2 \), \( (M^4, \sigma, \psi) \) is equivariantly symplectomorphic to either \( (\mathbb{C}P^2, \lambda \cdot \sigma_{\text{FS}}) \) or a product \( (\mathbb{C}P^1 \times \mathbb{C}P^1, \lambda \cdot (\sigma_{\text{FS}} \oplus \sigma_{\text{FS}})) \) (where \( \sigma_{\text{FS}} \) denotes the Fubini-Study form and these manifolds are equipped with the standard actions of \( T^2 \)), or
- if \( n = 1 \) or \( n > 2 \), \( (M^{2n}, \sigma, \psi) \) is equivariantly symplectomorphic to \( (\mathbb{C}P^n, \lambda \cdot \sigma_{\text{FS}}) \) (where \( \sigma_{\text{FS}} \) denotes the Fubini-Study form and this manifold is equipped with the standard action of \( T^n \)).

In a follow up paper [61], Pelayo and I proved that the space of equivariant ball packings of a symplectic-toric manifold admits the structure of a convex polytope. Using standard techniques from convex analysis, and in particular the Brunn-Minkowski inequality, we were able to understand the density function well enough to prove the following contrasting theorem:

Theorem 7.5. Let \( S^{2n} \) denote the set of equivalence classes of \( 2n \)-dimensional symplectic-toric manifolds and let \( \Omega_{2n} : S^{2n} \to (0, 1] \) be the maximal density function. Then \( \Omega^{-1}(\{x\}) \) is uncountable for all \( x \in (0, 1) \).

Problems about homogenous spaces. The geometry of general Riemannian homogenous spaces is a newer interest of mine.

Isospectral rigidity of left-invariant metrics. In recent joint work with C. Sutton [71], we considered the problem of determining left-invariant metrics on a compact simple three-dimensional Lie group via spectral invariants.

Let \( G \) denote one of the Lie groups \( SU(2) \) or \( SO(3) \).

Theorem 7.6. If two left-invariant Riemannian metrics on \( G \) have the same curvature tensor and the same volume, then they are isometric.

Note that the curvature tensor alone does not determine left-invariant metrics on \( G \) up to isometry [50].

Theorem 7.7. If two left-invariant Riemannian metrics on \( G \) have the same first four heat invariants, then they are isometric. In particular, left-invariant metrics on \( G \) are spectrally rigid in the class of left-invariant metrics.

Theorem 7.7 generalizes earlier work of C. Sutton, where he uses wave trace methods to prove spectral rigidity amongst the naturally reductive (Berger spheres) metrics on \( SU(2) \) and \( SO(3) \). Theorem 7.7 contrasts with results of Proctor and Schueth [64, 77] where it is shown that \( SO(n) \) admits nontrivial isospectral families.
of left-invariant metrics for \( n \) sufficiently large. These are not deformations of a bi-invariant metric since the bi-invariant metric is spectrally isolated amongst left-invariant metrics by work of Gordon, Schueth, and Sutton [36]. Our original proof of Theorem 7.7, while essentially elementary (polynomial algebra and calculus), was computationally quite involved. We are currently working on giving a cleaner and shorter argument, exploiting additional symmetry in the problem. We have preliminary results supporting the following:

**Conjecture 7.1.** Two compact Riemannian homogeneous three-manifolds are isospectral if and only if they are isometric.

Of course, this is not true for locally homogeneous manifolds.

**Ricci Signature and Rank of Fundamental Group.** In [87], J. Wolfson introduced the curvature condition \( k \)-positive Ricci for a Riemannian manifold: for each \( p \in M \), the sum of the smallest \( k \) eigenvalues of \( \text{Ric}_p \) is positive.

Known compact manifolds with 2-positive Ricci include manifolds with positive Ricci, Riemannian products of the circle with manifolds of positive Ricci, and connect sums of such manifolds. This motivates his conjecture:

**Conjecture 7.2.** If \( M \) is a closed \( n \)-manifold with 2-positive Ricci, then \( \pi_1 M \) is a virtually free group.

Wolfson and I have spent a fair amount of time discussing this problem without much concrete to show. Nevertheless, we can verify the conjecture in the class of compact homogeneous Riemannian manifolds, proving the more general:

**Theorem 7.8.** Let \( M \) be a compact Riemannian homogeneous space with \( \text{Ric} \) having \( n_p \) positive, \( n_z \) zero, and \( n_n \) negative eigenvalues. Then \( \pi_1(M) \) has free abelian rank at most \( n_z + n_n \).

The proof of Theorem 7.8, while not written, is mainly algebraic and not particularly enlightening. We are working to resolve Conjecture 7.2 for the larger class of compact locally homogeneous spaces, a task that we hope will shed more light on the geometric aspects of the conjecture.

**One dimensional Riemannian foliations for left-invariant metrics on \( SU(2) \).** In the process of completing the joint work with R. Shankar and R. Spatzier described in Section 2, we also started another project concerning Riemannian foliations.

Consider \( SU(2) \) endowed with a left-invariant metric \( g \). A one dimensional foliation \( \mathcal{F} \) of \( (SU(2), g) \) is Riemannian if its leaves are locally equidistant, or equivalently, if the plane field distribution orthogonal to \( \mathcal{F} \) is totally geodesic. Examples of Riemannian foliations include the orbit foliations of an isometric flow, the so-called homogeneous foliations. On round three-spheres, i.e. \( SU(2) \) with a bi-invariant metric, all the one dimensional Riemannian foliations are homogeneous by more general work of Gromoll and Grove [30]. One can ask if this property characterizes the bi-invariant metrics amongst the left-invariant metrics. We can prove the following:

**Theorem 7.9.** If \( g \) is a generic left-invariant Riemannian metric on \( SU(2) \), then \( (SU(2), g) \) admits a one dimensional Riemannian foliation that is not homogeneous.

In Theorem 7.9, the meaning of “generic” is that \( g \) is not bi-invariant nor belongs to the one parameter family of Berger metrics. We still cannot determine whether
or not Berger spheres can admit one dimensional Riemannian foliations that are not homogeneous.

References

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