GROUP REPRESENTATIONS AND
SYMMETRIC FUNCTIONS

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I. Group representations

II. Representations of the symmetric group, $\mathfrak{S}_n$

III. Associated combinatorics

IV. Symmetric functions

Copies available at

http://www.mth.msu.edu/~sagan
References


I. Group representations: A. Modules

All groups $G$ will be finite and all vector spaces $V$ will be over $\mathbb{C}$.

A matrix representation (rep) of a group $G$ is a group homomorphism

$$X : G \rightarrow GL_d(\mathbb{C}).$$

A $G$-module is a vector space $V$, $\dim V = d$, with a group homomorphism

$$\rho : G \rightarrow GL(V).$$

This gives a linear action of $G$ on $V$: $gv = \rho(g)v$.

The parameter $d$ is called the degree or dimension of the rep. We will freely go between matrix rep's and $G$-modules.

The group algebra is the $G$-module

$$\mathbb{C}[G] = \{ \sum_{g \in G} c_g g \mid c_g \in \mathbb{C} \}$$

with action $gh = k$ if $gh = k$ in $G$. The corresponding matrix rep in the basis $B = \{g \mid g \in G\}$ is called the (left) regular rep. The corresponding matrices $X(g)$ are permutation matrices (cf. Cayley’s Theorem).
Ex. Every group $G$ has the trivial rep $X^{\text{tri}}$

$$X^{\text{tri}}(g) = (1) \quad \text{for all } g \in G.$$ 

A module for this rep is $V$ with dim $V = 1$ and

$$gv = v \quad \text{for all } g \in G, v \in V.$$

Ex. For a cyclic group $G = \{g, g^2, \ldots, g^n = \epsilon\}$ any 1-dim rep would have $X(g) = (c)$ where

$$(c^n) = X(g^n) = X(\epsilon) = (1).$$

So $c$ is an $n$th root of 1 and all such $n$th roots give 1-dim rep’s.

If $n = 2$ then the group algebra is $\mathbb{C}[G] = \{c_1\epsilon + c_2g\}$ with action $g\epsilon = g, gg = \epsilon$. So the left regular rep is

$$X(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Changing basis to $\{\epsilon + g, \epsilon - g\}$ gives an equivalent rep

$$Y(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is a direct sum of the rep’s for $\sqrt{1} = \pm 1$.  

4
If $G$ acts on a set $S$ then one obtains a representation by linearly extending to the vector space

$$\mathbb{C}[S] = \{ \sum_{s \in S} c_s s \mid c_s \in \mathbb{C} \}.$$ 

The basis $S$ gives a rep by permutation matrices.

**Ex.** Given any group $G$, a subgroup $H \leq G$, and a set of all distinct left cosets

$$S = \{ t_1H, \ldots, t_lH \}$$

there is an action $gt_jH = t_iH$ if $gt_jH = t_iH$. The module $\mathbb{C}[S]$ is called a *coset rep*. If $H = G$ (resp. $H = \{e\}$) then it’s the trivial (resp. regular) rep.

**Ex.** The symmetric group $\mathfrak{S}_n$ acts by definition on

$$S = \{1, 2, \ldots, n\}.$$ 

The corresponding module $\mathbb{C}[1, \ldots, n]$ is the *defining rep*. If $n = 2$ then $(1, 2)1 = 2, (1, 2)2 = 1$ so

$$X(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X((1, 2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Also $\mathfrak{S}_n$ has the 1-dim *sign rep*

$$X(\pi) = (\text{sgn } \pi) \quad \text{sgn } \pi = \begin{cases} +1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd.} \end{cases}$$
B. Reducibility and Maschke’s Theorem

A submodule $W$ of $G$-module $V$, $W \leq V$, is a subspace closed under $G$’s action. Every $G$-module has trivial submodules $W = \{0\}, V$. Module $V$ is irreducible (irr) or an irrep if it has no non-trivial submodules. Every 1-dim module is an irrep.

**Ex.** The group algebra $\mathbb{C}[G]$ has submodule

$$W = \mathbb{C}[\sum_{g \in G} g]$$

since $h \sum_g g = \sum_g h g$. This $W$ gives the trivial rep.

If $G = \mathfrak{S}_n$ then we can get the sign rep with

$$U = \mathbb{C}[\sum_{\pi \in \mathfrak{S}_n} (\text{sgn } \pi)\pi].$$

**Ex.** If $G = \mathfrak{S}_n$ and $V = \mathbb{C}[1, \ldots, n]$ then

$$W = \mathbb{C}[1 + 2 + \cdots + n]$$

is a submodule for the trivial rep. Consider the inner product on $V$: $\langle i, j \rangle = \delta_{i,j}$ (Kronecker $\delta$). Then

$$W^\perp = \{ \sum c_i i \mid \sum c_i = 0 \}$$

is also a submodule and $V = W \oplus W^\perp$ with $W, W^\perp$ irr. (Clear for $W$, not for $W^\perp$.)
A $G$-module $V$ is completely reducible if

$$V = W^{(1)} \oplus \ldots \oplus W^{(k)}$$

where each $W^{(i)}$ is irr.

**Theorem 1 (Maschke)** If $G$ is finite then every complex $G$-module $V$ is completely reducible.

**Proof.** If $V$ is irr, we are done. If not, let $W$ be a non-trivial submodule. Pick a basis for $V$ $\mathcal{B} = \{v_1, \ldots, v_d\}$ with corresponding inner product $\langle v_i, v_j \rangle = \delta_{i,j}$. Now define another inner product

$$\langle v, w \rangle' = \sum_{g \in G} \langle gv, gw \rangle$$

which is $G$-invariant:

$$\langle hv, hw \rangle' = \sum_{g \in G} \langle ghv, ghw \rangle = \sum_{g \in G} \langle gv, gw \rangle = \langle v, w \rangle'.$$

Now $W^\perp$ (with respect to $\langle \cdot, \cdot \rangle'$) is a submodule since if $v \in W^\perp$, $w \in W$, and $g \in G$ then

$$\langle gv, w \rangle' = \langle v, g^{-1}w \rangle' = 0.$$

So $V = W \oplus W^\perp$ and done by induction on dim $V$. □

**Note:** 1. Maschke may not be true if $|G| = \infty$ or the field is different from $\mathbb{C}$.

2. Henceforth we can just concentrate on irreps.
C. G-homomorphisms and Schur's Lemma

A G-homomorphism (hom) of G-modules \( V, W \) is a linear map \( \theta : V \to W \) such that for all \( g \in G, v \in V \)

\[
\theta(gv) = g\theta(v).
\]

A bijective \( \theta \) is called a G-isomorphism (iso) and then \( V, W \) are G-equivalent (equiv), \( V \cong W \). Turning everything into matrices

\[
TX(g)v = Y(g)Tv \quad \text{for all } g \in G, v \in \mathbb{C}^d
\]

\[
\Rightarrow TX(g) = Y(g)T \quad \text{for all } g \in G
\]

\[
\text{def } TX = YT.
\]

Ex. Let \( V = \mathbb{C}[v] \) be the trivial rep and \( W = \mathbb{C}[G] \) be the group algebra. Then a G-hom is \( \theta : V \to W \) defined by

\[
\theta(v) = \sum_{g \in G} g.
\]

Ex. Let \( G = S_2 \), let \( V = \mathbb{C}[1, 2] \) be the defining rep and \( W = \mathbb{C}[\epsilon, (1, 2)] \) be the group algebra. Then \( \theta : V \to W \) by \( 1 \mapsto \epsilon, \ 2 \mapsto (1, 2) \) is an \( S_2 \)-iso, e.g.,

\[
\theta((1, 2)2) = \theta(1) = \epsilon = (1, 2)(1, 2) = (1, 2)\theta(2).
\]
Lemma 2 (Schur) If $V, W$ are irreducible modules and $\theta : V \to W$ is a $G$-homomorphism then either

1. $\theta$ is a $G$-isomorphism or

2. $\theta$ is the zero map.

Proof. Since $\theta$ is a $G$-hom, ker $\theta$ and im $\theta$ are $G$-submodules of $V$ and $W$, respectively. Since $V, W$ are irr, ker $\theta = \{0\}$ or $V$ and im $\theta = \{0\}$ or $W$. If ker $\theta = \{0\}$ and im $\theta = W$ then $\theta$ is a $G$-iso. All other cases lead to the zero map. ■

Schur’s Lemma is valid for infinite groups and arbitrary fields. For $\mathbb{C}$ more is true.

Corollary 3 If $X$ is an irreducible matrix representation (irrep) of $G$ over $\mathbb{C}$ and $T$ commutes with $X$ then $T = cI$, $c \in \mathbb{C}$.

Proof. Let $c$ be an eigenvalue of $T$. Then

$$TX = XT \quad \Rightarrow \quad (T - cI)X = X(T - cI).$$

By Schur, $T - cI$ is invertible or zero and the former can’t happen by the choice of $c$. ■
D. The endomorphism algebra

A $G$-module $V$ has endomorphism algebra

$$\text{End } V = \{ \theta : V \to V \mid \theta \text{ is a } G\text{-homomorphism} \}.$$  

For a $d$-dim matrix representation $X$ this becomes

$$\text{End } X = \{ T \in \text{Mat}_d \mid TX = XT \}.$$  

To describe $\text{End } X$ we use block matrix operations

$$S \oplus T = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}, \quad S \otimes T = (S_{i,j} T) \text{ where } S = (S_{i,j}).$$

Suppose that $X$ decomposes as

$$X = X^{(1)} \oplus X^{(2)} \oplus \ldots \oplus X^{(l)}$$

where the $X^{(i)}$ are irr. Let $T = (T_{i,j}) \in \text{End } X$ have the same block form. Then $XT = XT$ implies

$$T_{i,j} X^{(j)} = X^{(i)} T_{i,j} \text{ so}$$

$$T_{i,j} = \begin{cases} 0 & \text{if } X^{(i)} \not\cong X^{(j)} \text{ (Schur)} \\ c_{i,j} I & \text{if } X^{(i)} \cong X^{(j)} \text{ (Cor)}. \end{cases}$$

Renaming the irreps to collect equiv ones and letting $d_i = \text{dim } X^{(i)}$

$$X = \bigoplus_{i=1}^k m_i X^{(i)} \Rightarrow$$

$$\text{End } X = \{ \bigoplus_{i=1}^k (M_{m_i} \otimes I_{d_i}) \mid M_{m_i} \in \text{Mat}_{m_i} \forall i \}.$$
Otherwise put

$$\text{End } X \cong \bigoplus_{i=1}^{k} \text{Mat}_{m_i}.$$  

The center $Z_{\text{Mat}_m} = \{c I \mid c \in \mathbb{C}\}$ and so

$$Z_{\text{End } X} = \{\bigoplus_{i=1}^{k} c_i I_{m_i d_i} \mid c_i \in \mathbb{C} \text{ for all } i\} \cong \text{Diag}_k,$$

where Diag$_k$ are the diagonal matrices in Mat$_k$.

Summarizing and taking dimensions:

**Theorem 4** Let $X$ be a matrix rep of $G$ with

$$X = m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)}$$

where the $X^{(i)}$ are inequiv, irr and with dimensions $\dim X^{(i)} = d_i$. Then

1. $\text{End } X \cong \bigoplus_{i=1}^{k} \text{Mat}_{m_i}$,

2. $Z_{\text{End } X} \cong \text{Diag}_k$,

3. $\dim X = m_1 d_1 + m_2 d_2 + \cdots + m_k d_k$,

4. $\dim(\text{End } X) = m_1^2 + m_2^2 + \cdots + m_k^2$,

5. $\dim Z_{\text{End } X} = k$. 

11
E. Group characters and inner products

Matrix rep $X$ has character (char) $\chi : G \to \mathbb{C}$ where

$$\chi(g) = \text{tr} X(g).$$

A $G$-module also has a unique character since any two bases give conjugate matrix reps.

Ex. If $\dim X = 1$ then $\chi$ is called a linear char and

$$\chi(gh) = \text{tr} X(gh) = \text{tr} X(g) \text{tr} X(h) = \chi(g) \chi(h).$$

Ex. If $V = \mathbb{C}[G]$ (regular rep) then the char is

$$\chi^{\text{reg}}(g) = |\{h : gh = h\}| = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{else.} \end{cases}$$

Ex. If $V = \mathbb{C}[1, \ldots, n]$ (defining rep of $S_n$), then

$$\chi^{\text{def}}(\pi) = \text{number of fixed points of } \pi.$$

Proposition 5 Let group $G$ have matrix representation $X$ with $\dim X = d$ and character $\chi$.

1. $\chi(e) = d$,

2. If $K$ is a conjugacy class: $g, h \in K \Rightarrow \chi(g) = \chi(h),$

3. If rep $Y$ has char $\psi$: $X \cong Y \Rightarrow \chi = \psi.$
Character $\chi$ is a *class function* since it is constant on conjugacy classes $K$. Let $\chi_K = \chi(g), g \in K$. The *character table* of $G$ has rows indexed by the irreps ($\chi_{\text{tri}}$ first) columns indexed by the conjugacy classes ($\{\varepsilon\}$ first) and entries $\chi_K$. It is square.

**Ex.** If $G = \mathbb{S}_3$ then we have

<table>
<thead>
<tr>
<th></th>
<th>${\varepsilon}$</th>
<th>${(1, 2); (1, 3); (2, 3)}$</th>
<th>${(1, 2, 3); (1, 3, 2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{\text{tri}}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{\text{sgn}}$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{\text{mys}}$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

The *inner product* of $\chi, \psi : G \to \mathbb{C}$ is

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} = \frac{1}{|G|} \sum_K |K| \chi_K \overline{\psi_K}.$$

If $G$-module $V$ has char $\psi$ then an orthonormal basis for $V$ with respect to a $G$-invariant inner product on $V$ gives matrices for $\psi$ which are unitary and

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1}).$$

If $G = \mathbb{S}_n$ then $g$ and $g^{-1}$ are conjugate and so

$$\langle \chi, \psi \rangle = \frac{1}{n!} \sum_K |K| \chi_K \overline{\psi_K}.$$
Theorem 6 (Character relations, the 1st kind)
If $\chi, \psi$ are irreducible characters of a group $G$ then

$$\langle \chi, \psi \rangle = \delta_{\chi,\psi}. \quad (*)$$

Proof sketch. Let $\chi, \psi$ come from reps $X, Y$. Let
$$Z = (z_{i,j}) \text{ and } W = |G|^{-1}\sum_{g \in G} X(g)ZY(g^{-1}).$$
Then $XW = WY$ and by Schur's Lemma

$$W = \begin{cases} 
0 & \text{if } X \not\cong Y, \\
\text{cI} & \text{if } X \cong Y.
\end{cases}$$

Since this is true for all $Z$, one can get equations relating the entries of $X$ and $Y$ giving $(*)$. ■

Corollary 7 Let $X \cong \bigoplus_{i=1}^{k} m_i X^{(i)}$ where the $X^{(i)}$ are pairwise inequiv with char's $\chi^{(i)}$.

1. $\chi = m_1\chi^{(1)} + m_2\chi^{(2)} + \cdots + m_k\chi^{(k)},$
2. $\langle \chi, \chi^{(i)} \rangle = m_i,$
3. $\langle \chi, \chi \rangle = m_1^2 + m_2^2 + \cdots + m_k^2,$
4. $X \text{ is irr } \Leftrightarrow \langle \chi, \chi \rangle = 1 \text{ (use 3),}$
5. If $Y$ has char $\psi$ then $X \cong Y \Leftrightarrow \chi = \psi \text{ (use 2)}.
Ex. Let $G = G_3$ and $V = \mathbb{C}[1, 2, 3]$ (defining rep) with char $\chi = \chi^{\text{def}}$. Then

$$\chi(\pi) = \text{number of fixed points of } \pi$$

$$\chi(\epsilon) = 3, \ \chi((1, 2)) = 1, \ \chi((1, 2, 3)) = 0.$$ 

Also

$$\chi = m_1\chi^{\text{tri}} + m_2\chi^{\text{sgn}} + m_3\chi^{\text{mys}} \quad \text{where}$$

$$m_1 = (1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 1)/3! = 1$$

$$m_2 = (1 \cdot 3 \cdot 1 + 3 \cdot 1(-1) + 2 \cdot 0 \cdot 1)/3! = 0.$$ 

So

$$\chi = \chi^{\text{tri}} + m_3\chi^{\text{mys}}.$$ 

Consider the character

$$\psi = \chi - \chi^{\text{tri}},$$

$$\psi(\epsilon) = 2, \ \psi((1, 2)) = 0, \ \psi((1, 2, 3)) = -1.$$ 

Then $\psi$ is irreducible since

$$\langle \psi, \psi \rangle = (1 \cdot 2^2 + 3 \cdot 0^2 + 2(-1)^2)/3! = 1.$$ 

So $m_3 = 1$ and $\chi^{\text{mys}} = \psi$ giving the complete table

<table>
<thead>
<tr>
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</tr>
<tr>
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<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{\text{mys}}$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Note: For general $n$, $\chi^{\text{def}} - \chi^{\text{tri}}$ is irreducible.
F. Decomposing the group algebra

**Proposition 8** Let $\mathbb{C}[G] = \bigoplus_i m_i V^{(i)}$ where the $V^{(i)}$ are a complete list of all inequiv irreps

1. $m_i = \dim V^{(i)}$ (so all irreps occur),

2. $\sum_i (\dim V^{(i)})^2 = |G|$, 

3. $\# \text{ of irreps} = \# \text{ of conjugacy classes } K \text{ of } G$.

**Proof sketch.** 1. Let $\chi = \chi^{\text{reg}}$. Then

$$m_i = \frac{\sum_{g \in G} \chi(g)\chi^{(i)}(g^{-1})}{|G|} = \frac{\chi(\epsilon)\chi^{(i)}(\epsilon)}{|G|} = \dim V^{(i)}.$$ 

2. Follows from 1. 3. $# \text{ of irreps} = \dim Z_{\text{End } \mathbb{C}[G]}$.

$\text{End } \mathbb{C}[G] = \{ \phi_V : \phi_V(w) = wv \} \cong \mathbb{C}[G]$.

Now $z \in Z_{\mathbb{C}[G]}$ iff $z = hzh^{-1}$ for all $h \in G$. So for each conjugacy class $K$ of $G$, $Z_{\mathbb{C}[G]}$ has a basis element

$$z_K = \sum_{k \in K} k.$$
Corollary 9  1. The character table of $G$ is square.

2. The irr characters $\chi$ of $G$ form an orthonormal basis for the space $R(G)$ of class functions on $G$.

3. (Character relations of the second kind) If $K, L$ are conjugacy classes of $G$ and $\chi$ is irreducible

$$\sum_{\chi} \chi_K \overline{\chi_L} = \frac{|G|}{|K|} \delta_{K,L}. $$

Proof. 1 and 2 follow from part 3 of the Proposition and the character relations of the first kind.

3. The relations of the first kind also give that the modified character table $U = (\sqrt{|K|} |G| \chi_K)$ has orthonormal rows, thus orthonormal columns.

Ex. We can find $\chi^{mys}$ for $S_3$ another way. By the Proposition, part 2,

$$1^2 + 1^2 + \chi^{mys}(e)^2 = 3! \Rightarrow \chi^{mys}(e) = 2. $$

The other two entries are found using the relations of the second kind. For example, taking $K = \{e\}$ and $L = \{(1,2), \ldots\}$

$$0 = 1 \cdot 1 + 1(-1) + 2\chi^{mys}_L \Rightarrow \chi^{mys}_L = 0.$$
G. Representations of products and subgroups

If $X, Y$ are matrix reps of $G, H$ respectively then the tensor product rep of $G \times H$ is

$$(X \otimes Y)(g, h) = X(g) \otimes Y(h).$$

**Proposition 10**

1. $X \otimes Y$ is a rep of $G \times H$. If $X, Y$ are irreps then so is $X \otimes Y$.

2. As $X^{(i)}, Y^{(j)}$ run over complete lists of inequiv irreps for $G, H$ resp, $X^{(i)} \otimes Y^{(j)}$ runs over a complete list of inequiv irreps for $X \otimes Y$.

3. If $X, Y, X \otimes Y$ have characters $\chi, \psi, \chi \otimes \psi$ resp then

$$(\chi \otimes \psi)(g, h) = \chi(g)\psi(h).$$

**Proof of 2.** Suppose that $X^{(i)}, Y^{(j)}$ have chars $\chi^{(i)}, \psi^{(j)}$ resp. Then inequivalence follows from

$$\langle \chi^{(i)} \otimes \psi^{(j)}, \chi^{(k)} \otimes \psi^{(l)} \rangle = \langle \chi^{(i)}, \chi^{(k)} \rangle \langle \psi^{(j)}, \psi^{(l)} \rangle = \delta_{i,k} \delta_{j,l} = \delta_{(i,j),(k,l)}.$$ 

For list completeness, just check we have the right number of irreps. Let $k(\cdot) = \#$ of conjugacy classes.

$$\# \text{ of irreps of } G \times H = k(G \times H) = k(G)k(H) = (\# \text{ of irreps of } G)(\# \text{ of irreps of } H).$$
If $H \leq G$ and $X$ is a rep of $G$ then the restriction of $X$ to $H$, $X \downarrow_H = X \downarrow_H^G$, is

$$X \downarrow_H (h) = X(h).$$

It is clear the $X \downarrow_H$ is a rep, but if $X$ is irr then $X \downarrow_H$ need not be. For example, if $X$ is the 2-dim irrep of $G_3$ and $H = \{e, (1, 2)\} := G_{\{1,2\}} \cong G_2$ then

$$X \downarrow_H \cong X^{\text{tri}} \oplus X^{\text{sgn}}.$$

If $Y$ is a rep of $H$ then $Y(g) := 0$ for $g \not\in H$ doesn’t give a rep. But if $G = \bigcup t_i H$ then the induction of $Y$ to $G$, $Y \uparrow^G = Y \uparrow^G_H$, has block matrices

$$Y \uparrow^G (g) = (Y(t_i^{-1}gt_j)).$$

**Ex.** Consider $1 \uparrow^G$ for the trivial char $1$ of $H$. Then

$$1(g) = \begin{cases} 1 & \text{if } g \in H, \\ 0 & \text{if } g \not\in H. \end{cases}$$

So

$$1(t_i^{-1}gt_j) = 1 \iff t_i^{-1}gt_j \in H \iff gt_jH = t_iH.$$ 

So $1 \uparrow^G$ equals the coset rep $\mathbb{C}[H]$ in the standard basis $\mathcal{H} = \{t_1H, \ldots, t_iH\}$ and so consists of permutation matrices. In general, $Y \uparrow^G$ consists of block permutation matrices.
**Proposition 11**
1. \( Y \uparrow^G \) is a representation of \( G \) which may be reducible even if \( Y \) is an irrep of \( H \).
2. Two transversals of \( H \) give equiv induced reps.
3. If \( Y, Y \uparrow^G \) have chars \( \psi, \psi \uparrow^G \) resp then
   \[
   \psi \uparrow^G (g) = \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1}gx).
   \]
4. (Frobenius Reciprocity) If \( \chi \) is a char of \( G \) then
   \[
   \langle \psi \uparrow^G, \chi \rangle = \langle \psi, \chi \downarrow H \rangle.
   \]

**Proof of 4.** We have
\[
\langle \psi \uparrow^G, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi \uparrow^G (g) \chi(g^{-1})
\]
\[
= \frac{1}{|G||H|} \sum_{x,g \in G} \psi(x^{-1}gx) \chi(g^{-1})
\]
\[
= \frac{1}{|G||H|} \sum_{x,y \in G} \psi(y) \chi(xy^{-1}x^{-1})
\]
\[
= \frac{1}{|G||H|} \sum_{x,y \in G} \psi(y) \chi(y^{-1})
\]
\[
= \frac{1}{|H|} \sum_{y \in H} \psi(y) \chi(y^{-1})
\]
\[
= \frac{1}{|H|} \sum_{y \in H} \psi(y) \chi(y^{-1}) = \langle \psi, \chi \downarrow H \rangle.
\]
H. The group determinant

Indeterminates \( \{ c_g | g \in G \} \) give the group matrix

\[
\Gamma = (c_{g^{-1}h})_{g,h \in G}.
\]

In the case \( G = \{ g, g^2, \ldots, g^n = \epsilon \} \), \( \Gamma \) is a circulant.

Ex. If \( n = 3 \) with rows and cols indexed \( \epsilon, g, g^2 \)

\[
\Gamma = 
\begin{pmatrix}
c_\epsilon & c_g & c_{g^2} \\
c_{g^2} & c_\epsilon & c_g \\
c_g & c_{g^2} & c_\epsilon
\end{pmatrix}
:=
\begin{pmatrix}
c_0 & c_1 & c_2 \\
c_2 & c_0 & c_1 \\
c_1 & c_2 & c_0
\end{pmatrix}.
\]

Theorem 12 (Frobenius) If the irreps \( G \) are \( X^{(i)} \), 
\( \dim X^{(i)} = d_i \), \( 1 \leq i \leq k \), then

\[
\det \Gamma = \prod_{i=1}^{k} \Delta_i^{d_i} \text{ with } \Delta_i := \left| \sum_{g \in G} X^{(i)}(g)c_g \right| \text{ irr.} \]

Corollary 13 \( |c_{j-i}| = \prod_{\zeta^n=1} (c_0+c_1\zeta+\cdots+c_{n-1}\zeta^{n-1}) \).

Ex. \[
\begin{vmatrix}
c_0 & c_1 \\
c_1 & c_0
\end{vmatrix} = c_0^2 - c_1^2 = (c_0 + c_1)(c_0 - c_1).
\]

Open Problem. Find a combinatorial proof of the corollary: The det counts \( \mathcal{G}_n \) with weight \( \text{wt}_1 \). The product counts \( \mathcal{F} = \{ f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \} \) with weight \( \text{wt}_2 \). Partition \( \mathcal{G}_n = \bigsqcup_i S_i \), \( \mathcal{F} = \bigsqcup_j F_j \) s.t. \( \sum_{f \in F_j} \text{wt}_2 f = 0 \) for certain \( F_j \) and for the rest there’s a weight preserving bijection with the \( S_i \).
II. Reps of $\mathfrak{S}_n$: A. Permutation modules

The number of irreps of $\mathfrak{S}_n$ is the number of conjugacy classes which is the same as the number of partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$, $\lambda \vdash n$, i.e.,

$$\lambda \in (\mathbb{Z}^+)^l$$

is weakly decreasing and $\sum_i \lambda_i = n$.

To every $\lambda$ is associated a Young subgroup

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1, \ldots, \lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}} \times \cdots$$

The corresponding coset rep $M^\lambda$ (for $1 \uparrow \mathfrak{S}_n^\lambda$) is called a permutation module. These are not irreducible, but we will find an ordering $\succ$ of partitions such that

$$M^\lambda = S^\lambda \bigoplus \bigoplus_{\mu \succ \lambda} K_{\mu \lambda} S^\mu$$

where the $S^\mu$ are irreps and the $K_{\mu \lambda}$ multiplicities.

To conveniently describe $M^\lambda$: The Ferrers diagram of $\lambda$ is the set of dots or cells

$$\lambda = \{ (i, j) \in (\mathbb{Z}^+)^2 \mid 1 \leq j \leq \lambda_i \}.$$ 

Ex.

$$(4, 4, 2) = \bullet \bullet \bullet \bullet = \begin{array}{|c|c|c|c|}
\hline
\bullet & \bullet & \bullet & \bullet \\
\hline
\bullet & \bullet & \bullet & \bullet \\
\hline
\end{array} \equiv \begin{array}{|c|c|}
\hline
2,3 \\
\hline
\end{array}.$$
A Young tableau of shape $\lambda$ or $\lambda$-tableau, written $t = t^\lambda$ or $\text{sh} \ t = \lambda$, is a bijection

$$t : \lambda \to \{1, 2, \ldots, n\}, \quad t_{i,j} := t(i, j).$$

A tabloid, $\{t\}$, is an equivalence class of tableaux with the same corresponding rows.

**Ex.** All tableaux of shape $(2, 1)$ are

$$\begin{align*}
1 & \ 2, \\
2 & \ 1, \\
1 & \ 3, \\
3 & \ 2, \\
3 & \ 1, \\
2 & \ 1, \\
2 & \ 1
\end{align*}$$

If $t$ is the first tableau in the list

$$\{t\} = \left\{ \begin{array}{c}
1 & 2, \\
2 & 1 \\
3 & 2
\end{array} \right\} = \begin{array}{c}
1 \\
2 \\
3
\end{array} = \begin{array}{c}
1 \\
2 \\
3
\end{array}.$$

A $\pi \in \mathfrak{S}_n$ acts on tableau $t = (t_{i,j})$ by $\pi t = (\pi t_{i,j})$ and thus acts on tabloids. With this action

$$M^\lambda = \mathbb{C}[\{t\} \mid \text{all } \lambda\text{-tabloids } \{t\}].$$

**Ex.** $\lambda = (n)$ gives the trivial rep

$$M^{(n)} = \mathbb{C}[\begin{array}{c}1 \\
2 \\
\ldots \\
n\end{array}].$$

$\lambda = (1, 1, \ldots, 1) := (1^n)$ gives the regular rep

$$M^{(1^n)} \cong \mathbb{C}[\mathfrak{S}_n].$$

$\lambda = (n-1, 1)$ gives the defining rep (ignore 1st row)

$$M^{(n-1,1)} \cong \mathbb{C}[1, 2, \ldots, n].$$
B. Orderings on partitions

For partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ of $n$, the dominance partial order, $\lambda \trianglerighteq \mu$, is

for all $i \geq 1$: $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$

and the lexicographic (lex) total order, $\lambda > \mu$, is

for some $i \geq 1$: $\lambda_i > \mu_i$ and $\lambda_j = \mu_j$ for $j < i$.

Ex.

$(3, 3) \triangleright (3, 2, 1) : 3 \geq 3, 3 + 3 > 3 + 2, \ldots$
$(3, 3), (4, 1, 1)$ incomp in $\triangleright : 3 < 4, 3 + 3 > 4 + 1.$
$(4, 1, 1) > (3, 3) : 4 > 3.$
$(3, 3) > (3, 2, 1) : 3 = 3, 3 > 2.$

**Proposition 14** 1. $\lambda \trianglerighteq \mu$ implies $\lambda \triangleright \mu$.

2. (Dominance Lemma, DL) If $\forall i$ entries of row $i$ of tableau $s^\mu$ are in different col’s of $t^\lambda$ then $\lambda \trianglerighteq \mu$.

**Proof.** 2. Sort each column of $t^\lambda$ so the entries in the first $i$ rows of $s^\mu$ lie in the first $i$ rows of $t^\lambda$.

$$\sum_{j \leq i} \lambda_j = \# \text{ elements in first } i \text{ rows of } t^\lambda \geq \# \text{ elements in first } i \text{ rows of } s^\mu = \sum_{j \leq i} \mu_j.$$
C. The irreducible Specht modules

If \( H \subseteq \mathcal{S}_n \) then let
\[
H^- = \sum_{\pi \in H} (\text{sgn} \, \pi) \pi \in \mathbb{C}[\mathcal{S}_n].
\]

If tableau \( t \) has columns \( C_1, \ldots, C_m \) then let
\[
C_t := \mathcal{S}_{C_1} \times \cdots \times \mathcal{S}_{C_m} \text{ (the column group)},
\]
\[
\kappa_t := C_t^- = \kappa_{C_1} \kappa_{C_2} \cdots \kappa_{C_m},
\]
\[
e_t := \kappa_t\{t\} \text{ (the polytabloid)}.
\]

**Ex.** If \( t = \begin{array}{ccc}
4 & 1 & 2 \\
3 & 5 & 2
\end{array} \) \( \text{then} \)
\[
C_t = \mathcal{S}_{\{3,4\}} \times \mathcal{S}_{\{1,5\}} \times \mathcal{S}_{\{2\}},
\]
\[
\kappa_t = \epsilon - (3, 4) - (1, 5) + (3, 4)(1, 5)
= (\epsilon - (3, 4))(\epsilon - (1, 5)),
\]
\[
e_t = \begin{array}{c}
\frac{4 \ 1 \ 2}{3 \ 5} - \frac{3 \ 1 \ 2}{4 \ 5} - \frac{4 \ 5 \ 2}{3 \ 1} + \frac{3 \ 5 \ 2}{4 \ 1}
\end{array}.
\]

**Lemma 15** If \( \pi \in \mathcal{S}_n \) and \( t \) is a tableau then
\[
\kappa_{\pi t} = \pi \kappa_t \pi^{-1} \quad \text{and} \quad e_{\pi t} = \pi e_t.
\]
Partition $\lambda$ has \textit{Specht module}

\[ S^\lambda = \mathbb{C}[e_t \mid \text{all } \lambda\text{-tableaux } t]. \]

\textbf{Ex. 1.} $\lambda = (n)$ gives the trivial rep: Any $(n)$-tableau $t$ has $e_t = \begin{array}{c} 1 \\ 2 \\ \cdots \\ n \end{array}$ so

\[ \pi e_t = e_{\pi t} = e_t. \]

2. $\lambda = (1^n)$ gives the sign rep: For any $t = t(1^n)$

\[ \pi e_t = \pi \mathfrak{S}_n \{t\} = (\text{sgn } \pi) e_t. \]

3. $\lambda = (n-1, 1)$: Abbreviate $t = t^\lambda$ to the 2nd row

\[ e_t = \begin{array}{c} i \\ \cdots \\ k \\ j \end{array} - \begin{array}{c} j \\ \cdots \\ k \\ i \end{array} = j - i, \]

\[ S(n-1, 1) = \mathbb{C}[j - i \mid 1 \leq i < j \leq n], \]

\[ = \{ \sum_{i=1}^{n} c_i i \mid \sum_{i=1}^{n} c_i = 0 \}. \]

A $G$-module $U$ is \textit{cyclic, generated by $u \in U$} if

\[ U = \mathbb{C}[gu \mid g \in G]. \]

\textbf{Corollary 16} $S^\lambda$ is cyclic generated by any $e_t \in S^\lambda$.

Define an $\mathfrak{S}_n$-invariant inner product on $M^\lambda$ by

\[ \langle \{t\}, \{s\} \rangle = \delta_{\{t\},\{s\}}. \]
Lemma 17 (Sign Lemma, SL) Let $H \leq \mathfrak{S}_n$.

1. $\pi \in H \Rightarrow \pi H^- = H^- \pi = (\text{sgn } \pi) H^-.$

2. $u, v \in M^\lambda \Rightarrow \langle H^- u, v \rangle = \langle u, H^- v \rangle.$

3. $(b, c) \in H \Rightarrow H^- = k(\varepsilon - (b, c))$ for some $k \in \mathbb{C}[\mathfrak{S}_n].$

4. $b, c$ in the same row of tableau $s$ and $(b, c) \in H \Rightarrow H^- \{s\} = 0.$

Corollary 18 I. If $\text{sh } t = \lambda, \text{sh } s = \mu$ with $\kappa_t \{s\} \neq 0$ then $\lambda \triangleright \mu$. If $\lambda = \mu$ then $\kappa_t \{s\} = \pm e_t.$

II. (James’ Submodule Theorem) If $U$ is a submodule of $M^\mu$ then $U \supseteq S^\mu$ or $U \subseteq S^{\mu\perp}$.

III. The $S^\mu, \mu \vdash n$, are all inequiv $\mathfrak{S}_n$-irreps over $\mathbb{C}$.

Proof. I. $b, c$ in the same row of $s \Rightarrow b, c$ not in the same col of $t$ (else $\kappa_t \{s\} = 0$ by SL4) $\Rightarrow \lambda \triangleright \mu$ (DL).

If $\lambda = \mu \Rightarrow \{s\} = \pi \{t\}$ for some $\pi \in \mathfrak{S}_n$ and by SL1

$$\kappa_t \{s\} = \kappa_t \pi \{t\} = (\text{sgn } \pi) \kappa_t \{t\} = \pm e_t.$$

II. If $u \in U$ and $t = t^\mu \Rightarrow \kappa_t u = ce_t$ for $c \in \mathbb{C}$ by I. If some $c \neq 0 \Rightarrow e_t \in U$ and $S^\mu \subseteq U$. Else use SL2 to show $U \subseteq S^{\mu\perp}$. 

27
D. The standard tableaux basis for $S^\lambda$

Tableau $t$ is standard if its rows and col’s increase.

Ex. \[ 1 \ 3 \ 4 \]
2 5

is standard; \[ 2 \ 3 \ 4 \]
1 5

is not.

**Theorem 19** A basis for $S^\lambda$ is

\[ \{ e_t \mid t \text{ a standard } \lambda\text{-tableau} \}. \]

**Independence.** A composition is a permutation of a partition. If \( \{t\} \) is a tabloid, for \( i \geq 1 \) let

\[ \{t\}^i = \text{tabloid of all entries } \leq i \text{ in } \{t\}, \]
\[ \lambda^i = \text{the shape of } \{t\}^i, \text{ a composition.} \]

Ex. If \( \{t\} = \frac{2}{1} \frac{3}{1} \) then

\[ \{t\}^1 = \frac{0}{1} \frac{1}{1} \]
\[ \{t\}^2 = \frac{2}{1} \frac{1}{1} \]
\[ \{t\}^3 = \frac{2}{1} \frac{3}{1} \]
\[ \lambda^1 = (0, 1) \]
\[ \lambda^2 = (1, 1) \]
\[ \lambda^3 = (2, 1). \]

**Dominance order on tabloids** is

\[ \{t\} \succ \{s\} \iff \lambda^i \succ \mu^i \quad \forall i. \]

**Proposition 20** 1. (Tabloid Dominance Lemma) If \( k < l \) and \( k \) is lower than \( l \) in \( \{t\} \) then \( (k, l)\{t\} \succ \{t\} \).

2. \( t \) standard and \( \{s\} \) appears in \( e_t \) \( \Rightarrow \{t\} \succ \{s\} \).

3. The standard \( e_t \) are independent.
Span. To show $e_t$ a lin comb of standard $e_s$ one can assume the col’s of $t$ increase. (Else $\exists \pi \in C_t$ with col’s of $\pi t$ increasing and $e_{\pi t} = (\text{sgn} \, \pi) e_t$.) If $t$ has row descent $a > b$, it suffices to find tableaux $s$ s.t.

1. $e_t = -\sum_s (\text{sgn} \, \pi_s) e_s$ where $\pi_s t = s$,
2. $[s] \triangleright [t]$ for all $s$, $[s] = \text{col tabloid}$.

$A$ (resp $B$) := entries of $t$ below $a$ (resp above $b$).

The $s$ are all tableaux gotten by permuting $A \cup B$ s.t. the elements of $A \cup B$ still increase in their col’s.

Ex. If $t = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{pmatrix}$ with $2 > 1$ $\Rightarrow$ $A = \{2, 3\}$, $B = \{1\}$,

$s_1 = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ $s_2 = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}$

$\pi_1 = (1, 3, 2)$ $\pi_2 = (1, 2)$

$e_t = -(e_{s_1} - e_{s_2})$. 

29
E. Young’s natural representation

The matrix rep \( X^\lambda \) for \( S^\lambda \) in the standard basis is Young’s natural rep. Since \((k, k+1), 1 \leq k < n\), generate \( S_n \) it suffices to compute \( M = X^\lambda((k, k+1)) \). If \( t \) is standard than to find \( M_{t,t} \) we must express \((k, k+1)e_t \) in the standard basis.

1. If \( k, k+1 \) in the same col of \( t \Rightarrow (k, k+1) \in C_t \)

\[
\therefore (k, k+1)e_t = -e_t.
\]

2. If \( k, k+1 \) in the same row of \( t \Rightarrow (k, k+1)t \) has row descent \( k+1 > k \)

\[
\therefore (k, k+1)e_t = e_t \pm \text{ other } e_s \text{ with } [s] \triangleright [t].
\]

3. Else \((k, k+1)t = t' \) where \( t' \) is standard

\[
\therefore (k, k+1)e_t = e_{t'}.
\]

**Ex.** If \( \lambda = (2, 1) \) then the standard tableaux are

\[
t_1 = \begin{array}{c}
1 \\
2 \\
3
\end{array} \quad \text{and} \quad t_2 = \begin{array}{c}
1 \\
3 \\
2
\end{array}.
\]

If \((k, k+1) = (1, 2)\) then

\[
(1, 2)e_{t_1} = \begin{array}{c}
2 \\
3 \\
1
\end{array} - \begin{array}{c}
1 \\
3 \\
2
\end{array} = -e_{t_1}.
\]

\((1, 2)e_{t_2} \) was essentially computed last slide.

\[
\therefore X^{(2,1)}((1, 2)) = \begin{pmatrix}
-1 & -1 \\
0 & 1
\end{pmatrix}.
\]
F. The Branching and Young Rules

Partition $\lambda$ has *inner corner* $(i, j) \in \lambda$ if

$$\lambda^- = \lambda \setminus (i, j)$$

is a partition, and *outer corner* $(i, j) \not\in \lambda$ if

$$\lambda^+ = \lambda \cup (i, j)$$

is a partition.

\[ \bullet \bullet \bullet \bullet \]

**Ex.** If $\lambda = \bullet \bullet$ then

\[ \lambda^- : \bullet \bullet \bullet \]

\[ \lambda^+ : \bullet \bullet \bullet \bullet \]

**Theorem 21 (Branching Rule)** *If* $\lambda \vdash n$ *then*

1. $S^\lambda \uparrow_{\mathcal{S}_{n-1}} \cong \bigoplus_{\lambda^-} S^{\lambda^-}$,

2. $S^\lambda \downarrow_{\mathcal{S}_{n+1}} \cong \bigoplus_{\lambda^+} S^{\lambda^+}$.

**Ex.** From the example above

$$S^{(4,2,2)} \uparrow_{\mathcal{S}_9} \cong S^{(5,2,2)} \oplus S^{(4,3,2)} \oplus S^{(4,2,2,1)}.$$
Tableau $T$ is called \textit{semistandard} if it has strictly increasing columns while its rows weakly increase. The \textit{content} of $T$, $\text{ct } T$, is the composition $\mu$ s.t.

$$\mu_i = \# \text{ of } i's \text{ in } T.$$ 

\textbf{Ex.} $T = \begin{array}{ccc} 1 & 1 & 4 \\ 2 & 4 \end{array}$ has content $\mu = (2, 1, 0, 2)$.

The \textit{Kostka numbers} are

$$K_{\lambda \mu} = \# \text{ of semistandard } T, \text{ shape } \lambda, \text{ content } \mu.$$  

\textbf{Theorem 22 (Young’s Rule)}

$$M^\mu \cong \bigoplus_{\lambda \unrhd \mu} K_{\lambda \mu} S^\lambda.$$ 

\textbf{Ex.} If $\mu = (2, 1, 1)$ then the possible $\lambda \unrhd \mu$ are

$\lambda : (2, 1, 1) \quad (2, 2) \quad (3, 1) \quad (4)$

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 2 \quad 3$$

$T : \quad 2 \quad 2 \quad 3 \quad 3 \quad 1 \quad 1 \quad 3 \quad 2$

So $M^{(2,1,1)} \cong S^{(2,1,1)} \oplus S^{(2,2)} \oplus 2S^{(3,1)} \oplus S^{(4)}$.

\textbf{Note:} For any $\mu$, $K_{\mu \mu} = 1 = K_{(n) \mu}$.
III. Combinatorics: A. Schensted’s algorithm

Let \( \text{SYT}(\lambda) \) be the set of standard \( \lambda \)-tableaux and

\[
f^\lambda := |\text{SYT}(\lambda)| = \dim S^\lambda.
\]

For any group with irreps \( V(i) \): \( \sum_i (\dim V(i))^2 = |G| \).

If \( G = \mathfrak{S}_n \) the formula can be proved combinatorially

\[
\sum_{\lambda \vdash n} (f^\lambda)^2 = n!
\]

**Proof.** Construct the Robinson-Schensted bijection

\[
\pi \xleftrightarrow{R-S} (P, Q)
\]

where \( \pi \in \mathfrak{S}_n \) and \( P, Q \in \text{SYT}(\lambda) \) for some \( \lambda \).

\( \pi \xrightarrow{R-S} (P, Q) \): *Insert* \( x \in \mathbb{Z}^+ \) into increasing tableau \( P \) to get increasing tableau \( P' \), \( r_x(P) = P' \), by

1. Let \( i := 1 \)
2. If \( x > \) every element of row \( i \) of \( P \), put it at the end of the row and stop.
3. Else exchange \( x \) and the smallest \( P_{i,j} > x \). (We say \( x \) *bumps* \( P_{i,j} \).) Set \( i := i + 1 \) and go to 2.

**Ex.** Suppose \( x = 2 \)

\[
\begin{align*}
P = & \quad 1 & 3 & \leftarrow & 2 & 1 & 2 & \quad 1 & 2 & 1 & 2 & = r_2(P) \\
& 4 & 5 & \leftarrow & 3 & 3 & 5 & \quad 3 & 5 & & & \leftarrow & 4 & 4
\end{align*}
\]
Now if $\pi = x_1 \ldots x_n$ then construct a sequence of pairs $(\emptyset, \emptyset) = (P_0, Q_0), \ldots, (P_n, Q_n) = (P, Q)$ by

$$
P_k = r_{x_k}(P_{k-1}),$$
$$Q_k = Q_{k-1} \uplus k \text{ with } k \text{ in } \text{sh } P_k \setminus \text{sh } P_{k-1}.
$$

**Ex.** $\pi = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
3 & 2 & 4 & 1 & 5
\end{array}$ $\xrightarrow{R-S} \begin{pmatrix}
1 & 4 & 5 & 1 & 3 & 5 \\
2 & & & 2 \\
3 & & & 4
\end{pmatrix}$ by

$P_k : \emptyset, 3, 2, 2 4, 1 4, 1 4 5 = P$

$Q_k : \emptyset, 1, 1, 1 3, 1 3, 1 3 5 = Q$

$(P, Q) \xrightarrow{R-S} \pi$: Delete $P_{i,j}, r_{(i,j)}^{-1}P = (P', x)$, by

1. Remove $x : = P_{i,j}$ from its row and set $i : = i - 1$.
2. While $i \geq 1$ exchange $x$ and the greatest $R_{i,j} < x$ and set $i : = i - 1$.

**Ex.** Do the Ex on the previous page backwards.

Starting with $(P, Q)$ we obtain the reverse sequence $(P_n, Q_n), \ldots, (P_0, Q_0)$ and $\pi = x_1 \ldots x_n$ by

$$
Q_{k-1} = Q_k \setminus k
$$

$$
(P_{k-1}, x_k) = r_{(i,j)}^{-1}P_k \text{ where } Q_{i,j} = k.
$$
B. Properties of Robinson-Schensted

If \( \pi^R \rightarrow^S (P, Q) \) then the \( P \)-tableau of \( \pi \) is \( P(\pi) = P \)
and the \( Q \)-tableau of \( \pi \) is \( Q(\pi) = Q \).

If \( \pi = x_1 \ldots x_n \) then \( \pi^r = x_n \ldots x_1 \).

A subsequence of \( \pi = x_1 \ldots x_n, \sigma \subseteq \pi \), is

\[
\sigma = x_{k_1}, x_{k_2}, \ldots, x_{k_m} \quad \text{with} \quad k_1 < k_2 < \ldots < k_m.
\]

**Proposition 23** 1. \( P(\pi^r) = P(\pi)^t \) (the transpose)
2. If \( \text{sh} \ P(\pi) = (\lambda_1, \ldots, \lambda_l) \) then

\[
\lambda_1 = \text{length of a longest increasing } \sigma \subseteq \pi,
\]

\[
l = \text{length of a longest decreasing } \sigma \subseteq \pi.
\]

3. If \( \pi^R \rightarrow^S (P, Q) \) then \( \pi^{-1} \rightarrow^S (Q, P) \).

4. \( \sum_{\lambda \vdash n} f^\lambda = \# \text{ of involutions in } \mathfrak{S}_n \).

**Proof.** 1. One can define column insertion \( c_y(P) \)
and prove \( r_x c_y(P) = c_y r_x(P) \). Then

\[
P(\pi^r) = r_{x_1} \cdots r_{x_n}(\emptyset) = r_{x_1} \cdots r_{x_{n-1}} c_{x_n}(\emptyset)
= c_{x_n} r_{x_1} \cdots r_{x_{n-1}}(\emptyset) = \ldots = c_{x_n} \cdots c_{x_1}(\emptyset) = P(\pi)^t.
\]

4. By 3: \( \pi^R \rightarrow^S (P, P) \) iff \( \pi = \pi^{-1} \). So

\[
\sum_{\lambda \vdash n} f^\lambda = \# \text{ of } P = \# \text{ of involutions } \pi.
\]
When does $P(\pi) = P(\sigma)$?

**Ex.** For $\mathfrak{S}_3$: $P(123) = 1\ 2\ 3$, $P(321) = (1\ 2\ 3)^t$, $P(213) = P(231) = \frac{1\ 3}{2}$, $P(132) = P(312) = \frac{1\ 2}{3}$.

$\pi, \sigma$ differ by a **Knuth transposition** if for $x < y < z$:

1. $\{\pi, \sigma\} = \{x_1 \ldots yxz \ldots x_n, \ x_1 \ldots yzx \ldots x_n\}$, or
2. $\{\pi, \sigma\} = \{x_1 \ldots zxy \ldots x_n, \ x_1 \ldots zyx \ldots x_n\}$.

Also $\pi, \sigma$ are **Knuth equivalent**, $\pi \equiv_K \sigma$, if

$$\pi = \pi_1, \pi_2, \ldots, \pi_k = \sigma$$

with $\pi_i, \pi_{i+1}$ differing by a Knuth transposition $\forall i$.

**Ex.** $2\ 1\ 3 \equiv_K 2\ 3\ 1$ and $1\ 3\ 2 \equiv_K 3\ 1\ 2$.

**Theorem 24 (Knuth)** $P(\pi) = P(\sigma) \iff \pi \equiv_K \sigma$.

**Proof sketch.** "$\Leftarrow$" Type 1 transposition: $x$'s (resp $z$'s) insertion path is weakly left (resp strictly right) of $y$'s so $P(\pi) = P(\sigma)$. Type 2: then $\pi^r, \sigma^r$ differ by type 1 and

$$P(\pi^r) = P(\sigma^r) \Rightarrow P(\pi)^t = P(\sigma)^t \Rightarrow P(\pi) = P(\sigma).$$
C. Schützenberger’s jeu de taquin

If $\mu \subseteq \lambda$ then one has the skew diagram

$$\lambda/\mu = \{(i, j) \mid (i, j) \in \lambda, (i, j) \notin \mu\}.$$  

**Ex.** If $\mu = (2, 1)$ and $\lambda = (4, 4, 1)$

$$\lambda/\mu = \begin{array}{|c|c|}
\hline
& \\
\hline
& \\
\hline
& \\
\hline
\end{array}$$

If $P$ is an increasing tableau, $\text{sh} \ P = \lambda/\mu$, a **backward slide** into an inner corner $c$ of $\mu$, $j^c(P) = P'$, is

While $c = (i, j)$ is not an inner corner of $\lambda$, exchange $c$ and the smaller of $P_{i+1,j}, P_{i,j+1}$.

**Ex.** If $c = (1, 2)$ then

$$P = \begin{array}{|c|c|}
\hline
& 1 5, & 1 5, & 1 3 5, & 1 3 5 \\
\hline
2 3 7 & 2 3 7 & 2 7 & \\
4 & 4 & 4 & 4 \\
\hline
\end{array} = j^c(P).$$

A **forward slide** into outer corner $d = (i, j)$ of $\lambda$, $j_d(P) = P'$, exchanges $d$ with the larger of the numbers $P_{i-1,j}, P_{i,j-1}$, etc. until an outer corner of $\mu$ is reached. Clearly if $j^c(P) = P'$ vacating $d$ then

$$j_dj^c(P) = P \quad \text{and} \quad j^cj_d(P') = P'.$$
Let $\delta_n = (n - 1, n - 2, \ldots, 1)$. Any $\pi = x_1 \ldots x_n$ has a $\delta_{n+1}/\delta_n$-tableau with $x_j$ in $(n - j + 1, j)$.

Ex. $\pi = 132$ has tableau $\pi = \begin{array}{c} 2 \\ 3 \\ 1 \end{array}$.

A backward slide sequence for $P = P_1$ is

$$(c_1, \ldots, c_l) \text{ with } P_{i+1} = j^{c_i}(P_i) \text{ defined } \forall i.$$ 

If $l = |\mu|$ where $\text{sh } P = \lambda/\mu$ let $j(P) := j^{c_1} \cdots j^{c_l}(P)$.

Ex. (cont) If $c_1 = (2, 1), c_2 = (1, 2), c_3 = (1, 1)$

$$\begin{array}{c} 2 \\ 3 \\ 1 \end{array}, \begin{array}{c} 2 \\ 1 \end{array}, \begin{array}{c} 2 \\ 1 \\ 3 \\ 3 \end{array} = j(\pi).$$

**Theorem 25** (Schützenberger) $j(\pi) = P(\pi)$.

**Proof sketch.** If $P$ has rows $R_1, \ldots, R_l$ then its row word is $\rho(P) = R_l R_{l-1} \ldots R_1$.

Ex. $P = \begin{array}{c} 1 \\ 2 \end{array} \begin{array}{c} 3 \\ 4 \end{array} \begin{array}{c} 5 \\ 6 \end{array} \begin{array}{c} 7 \end{array}$ has $\rho(P) = 2 4 6 1 3 5 7$.

It is easy to prove $P(\rho(P)) = P$. Furthermore if $P$ is skew and $P' = j^c(P)$ then $\rho(P') \overset{K}{=} \rho(P)$. So

$$\rho(j(\pi)) \overset{K}{=} \rho(\pi) = \pi \overset{\text{apply } P}{\Rightarrow} j(\pi) = P(\pi). \blacksquare$$
D. The hook formula

The hook and hooklength of \((i, j) \in \lambda\) are
\[
H_{i,j} = \{(i', j), (i, j') \in \lambda \mid i' \geq i, j' \geq j\}, \quad h_{i,j} = |H_{i,j}|.
\]
The arm length and leg length of the hook are
\[
a_{i,j} = |\{(i, j') \in \lambda \mid j' > j\}|, \quad l_{i,j} = |\{(i', j) \in \lambda \mid i' > i\}|.
\]
Ex. In \(\lambda = (4^2, 3, 1)\)
\[
H_{2,2} = \begin{array}{ccc}
\bullet & a & a \\
\cdot & \cdot & \cdot \\
\end{array}
\text{and } h_{2,2} = 4, \quad a_{2,2} = 2, \quad l_{2,2} = 1.
\]

**Theorem 26 (Frame-Robinson-Thrall)** If we have \(\lambda \vdash n\), then
\[
f^{\lambda} = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.
\]
Ex. \((3, 2) \vdash 5\) has hooklengths
\[
\begin{array}{ccc}
4 & 3 & 1 \\
2 & 1 \\
\end{array}
\]
So \(f^{(3,2)} = \frac{5!}{4\cdot3\cdot2\cdot1^2} = 5\) which agrees with
\[
123, \quad 124, \quad 125, \quad 134, \quad 135.
\]
\[
45, \quad 35, \quad 34, \quad 25, \quad 24.
\]

Show \( n! = f^\lambda \prod_{(i,j)} h_{i,j} \) with a bijection

\[ T \leftrightarrow (P, J) \]

where \( \text{sh} \, T = \text{sh} \, P = \text{sh} \, J = \lambda \), \( T \) is any Young tableau, \( P \) is standard, and

\[-l_{i,j} \leq J_{i,j} \leq a_{i,j} \quad \forall (i, j) \in \lambda.\]

\( T \rightarrow (P, J) \): If \( T \) is standard of shape \( \lambda/\mu \) and entry \( x \in \mathbb{Z}^+ \) is in \( c \) then \( j^c(T) \) has \( x \) moving in place of \( \bullet \) and terminating when it becomes standard.

Ex. If \( c = (1, 2) \) contains 6

\[
T = \begin{array}{cccc}
6 & 1 & 5, & 1 & 6 & 5, & 1 & 3 & 5 \\
2 & 3 & 7, & 2 & 3 & 7, & 2 & 6 & 7 \\
4 & 4 & 4 & 4
\end{array} = j^c(T).
\]

Lex order \( \lambda \)'s cells \( c_1 > c_2 > \ldots > c_n \). Define

\[ T = T_1, \ldots, T_n = P \quad \text{where} \quad T_k = j^{c_k}(T_{k-1}). \]

Define \( J_1, \ldots, J_n = J \) by \( J_1 = 0 \) and if \( j^{c_k} \) starts in \( c_k = (i,j) \) and ends in \( (i',j') \) then \( J_k = J_{k-1} \) except

\[
(J_k)_{i,l} = \begin{cases} 
(J_{k-1})_{i,l+1} + 1 & \text{for } j \leq l < j', \\
 i - i' & \text{for } l = j'.
\end{cases}
\]
Ex. For spacing purposes we use $\bar{I}$ for $-1$.

$$
T_1 = 645, 645, 645, 643, 623, 123 = P,
231, 213, 123, 125, 145, 456
$$

$$
J_1 = 000, 000, 000, 00\bar{I}, 0\bar{I}\bar{I}, 00\bar{I} = J.
000, 010, 200, 200, 200, 200
$$

$(P, J) \rightarrow T$: To reconstruct $(P, J) = (T_n, J_n), \ldots, (T_1, J_1) = (T, 0)$, assume $(T_k, J_k)$ has been constructed. The possible cells for $c_k = (i, j)$ in $T_k$ are

$$
P = \{(i', j') | i' \geq i, j' \geq j, (J_k)_{i,j'} \leq 0, i' = i - (J_k)_{i,j'}\}.
$$

Define $j_d$ for $d \in P$ by having the slide stop at $c_k$. (must prove well-defined) The code of $j_d$ replaces each move north (resp west) with $N$ (resp $W$) written in reverse order.

Ex. For $c_6 = (1, 1)$: $P = \{(1, 1), (1, 2), (2, 3)\}$ and

$$
j_{1,1} : \emptyset, \quad j_{1,2} : W, \quad j_{2,3} = NW W.
$$

Lex order the codes using $W < \emptyset < N$. Then

$$
T_{k-1} = j_d(T_k) \text{ where } d \in P \text{ has maximum code.}
$$

Also if $c_k = (i, j), d = (i', j')$ then $J_{k-1} = J_k$ except

$$
(J_{k-1})_{i,l} = \begin{cases} (J_k)_{i, l-1} - 1 & \text{for } j < l \leq j' \\ 0 & \text{for } l = j. \end{cases}
$$
E. The determinantal formula

Theorem 27 (Frobenius) If \((\lambda_1, \ldots, \lambda_i) \vdash n\) then
\[
f^\lambda = n! \det(1/(\lambda_i - i + j)!)\]
where the determinant is \(l \times l\) and \(1/r! = 0\) if \(r < 0\).

\[\text{Ex. } f^{(3,2)} = 5! \begin{vmatrix} 1/3! & 1/4! \\ 1/1! & 1/2! \end{vmatrix} = 5.\]

Proof. It suffices to show the determinant equals the hook formula. We have
\[
\lambda_i + l = h_{i,1} + i \quad \Rightarrow \quad \lambda_i - i + j = h_{i,1} - l + j.
\]
So every row of the determinant is of the form
\[
[\cdots \quad 1/(h-2)! \quad 1/(h-1)! \quad 1/h!].
\]
After factoring out \(\prod_i 1/h_{i,1}\) we get rows
\[
[\cdots \quad h(h-1) \quad h \quad 1]
\]
which by column operations can be turned into
\[
[\cdots \quad (h-1)(h-2) \quad h-1 \quad 1].
\]
Putting \(\prod_i 1/(h_{i,1}-1)!\) back in we get \(\prod_i 1/h_{i,1}\) times the det for \(\lambda\) with its first column removed, so we're done by induction.
IV. Symmetric functions: A. Bases

Let \( x = \{x_1, x_2, \ldots \} \) and also consider \( \mathbb{C}[[x]] \), the corresponding formal power series algebra. Then \( \pi \in \mathfrak{S}_n \) acts on \( f \in \mathbb{C}[[x]] \) by

\[
\pi f(x_1, x_2, \ldots) = f(x_{\pi 1}, x_{\pi 2}, \ldots), \quad \pi(m) := m, m > n.
\]

We say \( f \) is symmetric if

\[
\pi f = f, \quad \forall \pi \in \mathfrak{S}_n, \forall n.
\]

Each partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \) has an associated monomial symmetric function

\[
m_\lambda = m_\lambda(x) = \sum x_{i_1}^{\lambda_1} \cdots x_{i_l}^{\lambda_l}
\]

where the sum is over all distinct monomials that have exponents \( \lambda_1, \ldots, \lambda_l \).

Ex.

\[
m_{(2,2,1)} = x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2x_2^2x_4 + \cdots
\]

The algebra of symmetric functions is

\[
\Lambda = \Lambda(x) = \mathbb{C}[m_\lambda].
\]

Note: \( f = \prod_{i \geq 1} (1 + x_i) \) is symmetric but isn’t in \( \Lambda \). We have a grading by degree

\[
\Lambda = \bigoplus_{n \geq 0} \Lambda^n, \quad \dim \Lambda^n = p(n), \text{ the } \# \text{ of } \lambda \vdash n.
\]
\[ p_n := m(n) = \sum_{i \geq 1} x_i^n \quad \text{power sum}. \]
\[ e_n := m(1^n) = \sum_{i_1 < \ldots < i_n} x_{i_1} \cdots x_{i_n} \quad \text{elementary}. \]
\[ h_n := \sum_{\lambda \models n} m_\lambda = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \cdots x_{i_n} \quad \text{complete homo}. \]

\textbf{Ex.} \quad p_3 = x_1^3 + x_2^3 + x_3^3 + \cdots \]
\[ e_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + \cdots \]
\[ h_3 = x_1^3 + \cdots + x_1^2 x_2 + \cdots + x_1 x_2 x_3 + \cdots \]

\textbf{Proposition 28} \quad \text{We have the generating functions}

1. \( E(t) := \sum_{n \geq 0} e_n(x) t^n = \prod_{i \geq 1} (1 + x_i t). \)
2. \( H(t) := \sum_{n \geq 0} h_n(x) t^n = \prod_{i \geq 1} \frac{1}{1 - x_i t}. \)
3. \( P(t) := \sum_{n \geq 1} p_n(x) t^n = \ln \prod_{i \geq 1} \frac{1}{1 - x_i t}. \)

If \( f = p, e, \) or \( h \) and \( \lambda = (\lambda_1, \ldots, \lambda_l) \) let \( f_\lambda = \prod_i f_{\lambda_i}. \)

\textbf{Theorem 29} \quad \text{Three bases for } \Lambda^n \text{ are}

1. \( \{ e_\lambda \mid \lambda \models n \} \), 2. \( \{ h_\lambda \mid \lambda \models n \} \), 3. \( \{ p_\lambda \mid \lambda \models n \} \).

\textbf{Proof.} \ 1 \Rightarrow \text{XS2.} \quad |\{ h_\lambda \}| = p(n) \text{ so it suffices to show every } e_n \text{ is a polynomial in } h_k. \ \text{But } H(t)E(-t) = 1 \text{ and taking the coefficient of } t^n, \ n \geq 1,
\[
\sum_{k=0}^{n} (-1)^k h_{n-k} e_k = 0 \Rightarrow e_n = h_1 e_{n-1} - h_2 e_{n-2} + \cdots \]
B. Schur functions

For tableau $T$ let $x^T = x^\mu = x^{\mu_1} \cdots x^{\mu_m}$ where $T$’s content is $\mu = (\mu_1, \ldots, \mu_m)$. A Schur function is

$$s_\lambda(x) = \sum_T x^T$$

summed over all semistandard $T$ of shape $\lambda$. Note $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$.

**Ex.**

$T: \begin{array}{ccccccc}
1 & 1 & 1 & 2 & \cdots & 1 & 2 & 1 & 3 & \cdots \\
2 & 2 & \cdots & 3 & 2
\end{array}$

$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + \cdots + 2 x_1 x_2 x_3 + \cdots$

The **alternant** for $\lambda = (\lambda_1, \ldots, \lambda_l)$ is

$$a_\lambda = |x^\lambda_{i,j}|_{1 \leq i, j \leq l}.$$ 

If $\delta = (l - 1, l - 2, \ldots, 0)$ then $a_\delta = \text{Vandermonde}$. Let $\chi^\lambda$ be an irr character and $k_\mu$ be the size of a conjugacy class in $\mathfrak{S}_n$. Let $K_{\lambda \mu}$ be a Kostka number.

**Theorem 30** If $\lambda = (\lambda_1, \ldots, \lambda_l)$ then

1. $\{s_\lambda \mid \lambda \vdash n\}$ is a basis of $\Lambda^n$.
2. $s_\lambda = \sum_{\mu \subseteq \lambda} K_{\lambda \mu} m_\mu$.
3. $s_\lambda = \frac{1}{n!} \sum_{\mu \vdash n} k_\mu \chi^\lambda_\mu p_\mu$.
4. $s_\lambda(x_1, \ldots, x_l) = \frac{a_{\lambda+\delta}}{a_\delta}$.
5. (Jacobi-Trudi) $s_\lambda = |h_\lambda_{i-i+j}|_{1 \leq i, j \leq l}$.
**Proof of 5.** (Gessel-Viennot-Lindström) A lattice path in $\mathbb{Z}^2$ is $p = s_1, s_2, \ldots$ where each $s_i$ is a unit step $N$ or $E$. Label the $E$ steps by

$$N(s_i) = \text{(number of } N \text{ steps preceding } s_i) + 1.$$ 

$$p = \begin{array}{cccc}
\vdots & \vdots & \vdots & s_8 \\
\vdots & \vdots & \vdots & s_7 \\
2 & 3 & 3 & s_6 \\
2 & 2 & s_5 & s_4 \\
s_2 & s_3 & s_4 & . \\
s_1 & . & . & .
\end{array}$$

$$x^p = x_2^2 x_3^2.$$ 

If $p$ is from $(a, b)$ to $(c, d)$ write $(a, b) \xrightarrow{p} (c, d)$. Let

$$x^p := \prod_{s_i \in E \in p} x_{N(s_i)} \Rightarrow h_n = \sum_{(a,b) \xrightarrow{p}(a+n,\infty)} x^p.$$ 

Fix $(u_1, \ldots, u_l), (v_1, \ldots, v_l)$ & form $\mathcal{P} = (p_1, \ldots, p_l)$ where for all $i: u_i \xrightarrow{p_i} v_{\pi i}$ for some $\pi \in \mathcal{S}_l$. Let

$$x^\mathcal{P} := \prod_{i} x^{p_i} \quad \text{and} \quad \text{sgn } \mathcal{P} := \text{sgn } \pi.$$ 

$$\mathcal{P} = \begin{array}{cccc}
u_4 & v_3 & v_2 & v_1 \\
\vdots & \vdots & \vdots & \vdots \\
u_4 & u_3 & u_2 & u_1 \\
\end{array}$$

$$x^\mathcal{P} = x_2^4 x_3^2 x_4,$$

$$\text{sgn } \mathcal{P} = \text{sgn}(1, 2, 3)(4) = +1.$$
Given $\lambda = (\lambda_1, \ldots, \lambda_l)$ pick

$$u_i := (1 - i, 0) \quad \text{and} \quad v_i := (\lambda_i - i + 1, \infty) \implies h_{\lambda_i - i + j} = \sum_{u_j \rightarrow v_i} x^p \quad \text{and} \quad |h_{\lambda_i - i + j}| = \sum_{\mathcal{P}} (\text{sgn } \mathcal{P}) x^p.$$

Define a sign-reversing involution $\mathcal{P} \leftrightarrow \mathcal{P}'$ by

1. If $\mathcal{P}$ is non-$\cap$ then $\mathcal{P}' = \mathcal{P}$.
2. Else, let $(i, j)$ be the lex least pair s.t. $p_i \cap p_j \neq \emptyset$, and $w \in p_i \cap p_j$ be SW-most, so $\mathcal{P}' = (\mathcal{P} \setminus p_i, p_j) \cup p'_i, p'_j$.

$$p'_i := u_i \xrightarrow{p_i} w \xrightarrow{p_j} v_{\pi j} \quad \text{and} \quad p'_j := u_j \xrightarrow{p_j} w \xrightarrow{p_i} v_{\pi i}.$$ 

All terms in the det cancel except $\mathcal{P}$ for non-$\cap$ paths which correspond to semistandard $\lambda$-tableaux $T$.

$$\mathcal{P} = \begin{array}{cccc} \cdot & v_3 & & v_1 \\ \cdot & & v_2 & \\ 4 & 3 & \cdot & \\ \cdot & & & \\ u_3 & u_2 & u_1 & \cdot \end{array} \quad T = \begin{array}{ccc} 1 & 2 & 2 \\ \cdot & 3 & \cdot \\ 2 & 2 & \cdot \\ 2 & \cdot & \cdot \\ 4 & \cdot & \cdot \end{array}$$
C. Knuth’s algorithm

Theorem 31 (Littlewood) If \( y = \{y_1, y_2, \ldots \} \) then
\[
\sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \prod_{i,j \geq 1} 1/(1 - x_i y_j).
\]

Proof (Knuth). Want a wt-preserving bijection
\[ \pi \xleftarrow{R-S} \xrightarrow{K} (T, U) \]
where \( T, U \) are semistandard of the same shape,
\[ \text{wt}(T, U) = x^T y^U. \]
Furthermore, \( \pi \) is a generalized permutation: a 2-line array with entries in \( \mathbb{Z}^+ \) in lex order, and
\[ \text{wt} \pi = \prod x_j y_i \]
where the product is over all col \( \binom{i}{j} \in \pi \).

Ex. \( \pi = \begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 \\
2 & 3 & 3 & 1 & 2
\end{array} \), with \( \text{wt} \pi = x_1 x_2^2 x_3^2 y_1^3 y_2^2 \).

The bijection is now the same as R-S.

Ex. (cont)
\[ T_i : \begin{array}{cccccc}
\phi & 2 & 2 & 3 & 2 & 3 \\
\end{array} \]
\[ \begin{array}{cccccc}
1 & 3 & 3 & 1 & 2 & 3 \\
2 & 2 & 3
\end{array} = T, \]
\[ U_i : \begin{array}{cccccc}
\phi & 1 & 1 & 1 & 1 & 1 \\
\end{array} \]
\[ \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2
\end{array} = U. \]
D. The characteristic map

Let \( R^n = R(\mathcal{S}_n) \) (class functions) and \( R = \oplus_{n \geq 0} R^n \). The characteristic map, \( \text{ch} : R \rightarrow \Lambda \), linearly extends

\[
\text{ch}(\chi) := \frac{1}{n!} \sum_{\mu \vdash n} k_\mu \chi_\mu \, p_\mu \quad \text{where} \quad \chi \in R^n.
\]

If \( \chi^\lambda \) is an irr character then \( \text{ch}(\chi^\lambda) = s_\lambda \) so \( \text{ch} \) is a v.s. iso which becomes an isometry if we define

\[
\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}.
\]

Finally for \( \chi, \psi \) chars of \( \mathcal{S}_n, \mathcal{S}_m \) let

\[
\chi \cdot \psi = (\chi \otimes \psi) \uparrow_{\mathcal{S}_n \times \mathcal{S}_m}
\]

and extend linearly. Then we have

\[
\begin{align*}
\text{ch}(\chi \cdot \psi) &= \langle \chi \cdot \psi, p \rangle \\
&= \langle (\chi \otimes \psi) \uparrow_{\mathcal{S}_n \times \mathcal{S}_m}, p \rangle \\
&= \langle (\chi \otimes \psi), p \downarrow_{\mathcal{S}_n \times \mathcal{S}_m} \rangle \\
&= \frac{1}{n!m!} \sum_{\lambda \vdash n, \mu \vdash m} k_\lambda k_\mu \chi_\lambda \psi_\mu p_\lambda p_\mu \\
&= \text{ch}(\chi) \text{ch}(\psi).
\end{align*}
\]

**Theorem 32** The map \( \text{ch} : R \rightarrow \Lambda \) is an isomorphism of algebras.
E. The Littlewood-Richardson Rule

Word $R = r_1 \ldots r_n \in (\mathbb{Z}^+)^n$ is a lattice permutation (lp) if for all $R_i = r_1 \ldots r_i$ and all $j \in \mathbb{Z}^+$

number of $j$’s $\geq$ number of $j + 1$’s in $R_i$.

Such $R$ corresponds to a standard tableau $P$ by

if $r_i = j$ then put $i$ in row $j$ of $P$.

1 2 6

Ex. $R = 1 1 2 3 2 1 3 \longleftrightarrow P = 3 5 4 7$.

Theorem 33 (Littlewood-Richardson, L-R) If

\[ s_\lambda s_\mu = \sum_{\nu} c^{\nu}_{\lambda \mu} s_\nu \]

then $c^{\nu}_{\lambda \mu}$ is the number of semistandard $T$ such that

1. $sh T = \nu / \lambda$ and $ct T = \mu$,

2. the reverse row word $\rho(T)^r$ is an lp.

Ex. For $s(2)s(2,1)$

\[
T: \begin{array}{cccccccc}
\bullet & \bullet & 1 & 1, & \bullet & \bullet & 1, & \bullet & \bullet \\
2 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 2
\end{array}
\]

\[s(2)s(2,1) = s(4,1) + s(3,2) + s(3,1^2) + s(2^2,1)\]

The L-R rule generalizes both the Branching Rule (for $s_\lambda s(1)$) and Young’s Rule (for $s(l)s(m)$).
F. The Murnagham-Nakayama Rule

A rim hook, $H$, is a skew shape that's a lattice path. A rim hook tableau $T$ has rows and cols weakly increasing and all $i$'s in a rim hook for each $i \in T$.

Ex. $H = \begin{array}{ccc} & & \hline \end{array}$ and $T = \begin{array}{cccc} 1 & 1 & 1 & 2 & 4 \\ & 2 & 2 & 2 & 4 \\ 3 & 3 & 3 & 4 & 4 \end{array}$.

Rim hook $H$ has leg length

\[ l(H) = (\text{number of rows of } H) - 1 \]

and a rim hook tableau $T$ has sign

\[ \text{sgn } T = \prod_{H \in T} (-1)^{l(H)}. \]

Ex. (cont) $l(H) = 2$, $\text{sgn } T = (-1)^{0+1+0+2} = -1$.

**Theorem 34 (Murnagham-Nakayama)** We have

\[ \chi^\lambda_\mu = \sum_{T} \text{sgn } T \]

sum over all rim hook tableaux, $\text{sh } T = \lambda$, $\text{ct } T = \mu$.■

Note $\chi^\lambda_{(1^n)} = f^\lambda$ is a special case.

Ex. For $\chi = \chi^{(2,1)}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>(1³)</th>
<th>(2,1)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>1 2, 1 3</td>
<td>1 1, 1 2</td>
<td>1 1</td>
</tr>
<tr>
<td></td>
<td>3 2</td>
<td>2 1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_\mu$</td>
<td>1 + 1 = 2</td>
<td>1 - 1 = 0</td>
<td>-1</td>
</tr>
</tbody>
</table>

51
G. Chromatic symmetric functions

A proper coloring of $G = (V, E)$ is $c : V \rightarrow \{1, \ldots, t\}$

$uv \in E \Rightarrow c(u) \neq c(v)$.

The chromatic polynomial of $G$ is

$P(G) = P(G, t) := \# \text{ of proper } c : V \rightarrow \{1, \ldots, t\}.$

Ex. If $G = v_2 \triangle v_3$ then

$P(G) = \prod_i (\# \text{ of } c(v_i)) = t(t - 1)(t - 2)$.

The chromatic symmetric function of $G$ is

$X(G) = X(G, x) = \sum_{\text{proper } c : V \rightarrow \mathbb{Z}^+} x_{c(v_1)} \cdots x_{c(v_n)}.$

Ex.

$G : \begin{array}{ccc}
1 & 2 & 1 \\
\downarrow & \downarrow & \downarrow \\
2 & 1 & 2 \\
\end{array}$

$X(G) = x_1^2 x_2 + x_1 x_2^2 + \cdots + 6 x_1 x_2 x_3 + \cdots$

Poset $P$ has incomparability graph $G = \text{inc } P$ with

$V = P$, \quad $E = \{uv \mid u, v \text{ incomparable in } P\}$

and is 3+1-free if it has no induced $\{a < b < c, d\}$.

**Conjecture 35 (Stanley-Stembridge)** If poset $P$ is 3+1-free and $X(\text{inc } P) = \sum_{\lambda} c_\lambda e_\lambda \Rightarrow c_\lambda \in \mathbb{Z}^+ \cup \{0\}$.

Gasharov has proved this with $e_\lambda$ replaced by $s_\lambda$. 

52
Acknowledgment. I would like to thank Shalom Eliahou for carefully reading these slides and pointing out a number of errata.