Pattern avoidance and quasisymmetric functions

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

September 23, 2014
Patterns and quasisymmetric functions

Main result

Questions
Outline

Patterns and quasisymmetric functions

Main result

Questions
Let $\mathcal{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. 

Say $\sigma \in \mathcal{S}_n$ avoids $\pi \in \mathcal{S}_k$ if no subsequence of $\sigma$ is order isomorphic to $\pi$.

Ex. $\sigma = 3125476$ avoids $\pi = 321$ since $\sigma$ has no decreasing subsequence of length 3.

If $\Pi$ is a set of permutations then let $\mathcal{S}_n(\Pi) = \{\sigma \in \mathcal{S}_n : \sigma$ avoids every $\pi \in \Pi\}$.

If $\sigma = a_1 \ldots a_n \in \mathcal{S}_n$ then let $\text{Des} \sigma = \{i \in [n-1] : a_i > a_{i+1}\}$, $\text{des} \sigma = |\text{Des} \sigma|$, $\text{maj} \sigma = \sum_{i \in \text{Des} \sigma} i$.

Dokos, Dwyer, Johnson, Selsor, and S studied the polynomials $M_n(\Pi; q, t) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\text{maj} \sigma} t^{\text{des} \sigma}$ for all $\Pi \subseteq \mathcal{S}_3$.

Alex Woo asked what would happen if you looked at analogous quasisymmetric functions.
Let $\mathfrak{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. Say $\sigma \in \mathfrak{S}_n$ avoids $\pi \in \mathfrak{S}_k$ if no subsequence of $\sigma$ is order isomorphic to $\pi$. 

Ex. $\sigma = 3125476$ avoids $\pi = 321$ since $\sigma$ has no decreasing subsequence of length 3.

If $\Pi$ is a set of permutations then let $\mathfrak{S}_n(\Pi) = \{\sigma \in \mathfrak{S}_n : \sigma \text{ avoids every } \pi \in \Pi\}$.

If $\sigma = a_1 \ldots a_n \in \mathfrak{S}_n$ then let $\text{Des } \sigma = \{i \in [n-1] : a_i > a_{i+1}\}$, $\text{des } \sigma = |\text{Des } \sigma|$, $\text{maj } \sigma = \sum_{i \in \text{Des } \sigma} i$.

Dokos, Dwyer, Johnson, Selsor, and S studied the polynomials $M_n(\Pi; q, t) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} q^{\text{maj } \sigma} t^{\text{des } \sigma}$ for all $\Pi \subseteq \mathfrak{S}_3$.

Alex Woo asked what would happen if you looked at analogous quasisymmetric functions.
Let $\mathcal{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. Say $\sigma \in \mathcal{S}_n$ avoids $\pi \in \mathcal{S}_k$ if no subsequence of $\sigma$ is order isomorphic to $\pi$.

Ex. $\sigma = 3125476$ avoids $\pi = 321$ since $\sigma$ has no decreasing subsequence of length 3.
Let $S_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. Say $\sigma \in S_n$ avoids $\pi \in S_k$ if no subsequence of $\sigma$ is order isomorphic to $\pi$.

**Ex.** $\sigma = 3125476$ avoids $\pi = 321$ since $\sigma$ has no decreasing subsequence of length 3.

If $\Pi$ is a set of permutations then let

$$S_n(\Pi) = \{ \sigma \in S_n : \sigma \text{ avoids every } \pi \in \Pi \}.$$
Let $S_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. Say $\sigma \in S_n$ avoids $\pi \in S_k$ if no subsequence of $\sigma$ is order isomorphic to $\pi$.

**Ex.** $\sigma = 3125476$ avoids $\pi = 321$ since $\sigma$ has no decreasing subsequence of length 3.

If $\Pi$ is a set of permutations then let

$$S_n(\Pi) = \{\sigma \in S_n : \sigma \text{ avoids every } \pi \in \Pi\}.$$

If $\sigma = a_1 \ldots a_n \in S_n$ then let

$$\text{Des } \sigma = \{i \in [n - 1] : a_i > a_{i+1}\},$$

$$\text{des } \sigma = |\text{Des } \sigma|,$$

$$\text{maj } \sigma = \sum_{i \in \text{Des } \sigma} i.$$
Let $\mathcal{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. Say $\sigma \in \mathcal{S}_n$ avoids $\pi \in \mathcal{S}_k$ if no subsequence of $\sigma$ is order isomorphic to $\pi$.

**Ex.** $\sigma = 3125476$ avoids $\pi = 321$ since $\sigma$ has no decreasing subsequence of length 3.

If $\Pi$ is a set of permutations then let

$$\mathcal{S}_n(\Pi) = \{\sigma \in \mathcal{S}_n : \sigma \text{ avoids every } \pi \in \Pi\}.$$ 

If $\sigma = a_1 \ldots a_n \in \mathcal{S}_n$ then let

$$\text{Des } \sigma = \{i \in [n - 1] : a_i > a_{i+1}\},$$

$$\text{des } \sigma = |\text{Des } \sigma|,$$

$$\text{maj } \sigma = \sum_{i \in \text{Des } \sigma} i.$$

Dokos, Dwyer, Johnson, Selsor, and S studied the polynomials

$$M_n(\Pi; q, t) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\text{maj } \sigma} t^{\text{des } \sigma}$$

for all $\Pi \subseteq \mathcal{S}_3$. 
Let $\mathcal{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. Say $\sigma \in \mathcal{S}_n$ avoids $\pi \in \mathcal{S}_k$ if no subsequence of $\sigma$ is order isomorphic to $\pi$.

**Ex.** $\sigma = 3125476$ avoids $\pi = 321$ since $\sigma$ has no decreasing subsequence of length 3.

If $\Pi$ is a set of permutations then let

$$\mathcal{S}_n(\Pi) = \{ \sigma \in \mathcal{S}_n : \sigma \text{ avoids every } \pi \in \Pi \}.$$

If $\sigma = a_1 \ldots a_n \in \mathcal{S}_n$ then let

$$\text{Des } \sigma = \{ i \in [n - 1] : a_i > a_{i+1} \},$$

$$\text{des } \sigma = | \text{Des } \sigma |,$$

$$\text{maj } \sigma = \sum_{i \in \text{Des } \sigma} i.$$

Dokos, Dwyer, Johnson, Selsor, and S studied the polynomials

$$M_n(\Pi; q, t) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\text{maj } \sigma} t^{\text{des } \sigma}$$

for all $\Pi \subseteq \mathcal{S}_3$. Alex Woo asked what would happen if you looked at analogous quasisymmetric functions.
Let \( x = \{x_1, x_2, \ldots \} \).
Let $x = \{x_1, x_2, \ldots \}$. The *symmetric functions*, $\text{Sym}_n$, are $f(x) \in \mathbb{C}[[x]]$ which are invariant under permutation of variables and homogeneous of degree $n$. 

**Bases for Sym$_n$** are indexed by *integer partitions* $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

Ex. $M(2,1) = x_2^2 + x_1x_2^2 + \cdots + x_1x_2x_3 + x_2x_3 + \cdots$.

**Quasisymmetric functions**, $\text{QSym}_n$, are invariant under order preserving variable substitution and homogeneous of degree $n$. Compositions $\alpha = (\alpha_1, \ldots, \alpha_k) |\alpha| = n$ index bases for $\text{QSym}_n$.

Ex. $M(1,2) = x_1x_2^2 + x_1x_2^3 + \cdots + x_1x_2x_3 + x_2x_3 + \cdots$.

Equivalently, bases for $\text{QSym}_n$ are indexed by sets $S \subseteq \mathbb{[n-1]}$.

The corresponding *fundamental quasisymmetric function* is $F_S = \sum x_{i_1}x_{i_2}^2\cdots x_{i_n}$ summed over $i_1 \leq i_2 \leq \cdots \leq i_n$ with $i_j < i_j + 1$ iff $j \in S$.

Ex. $n = 3$: $F_\{1\} = x_1x_2^2 + x_1x_2^3 + \cdots + x_1x_2x_3 + x_2x_3 + \cdots$.

Given a set of permutations $\Pi$, define $Q_n(\Pi) = Q_n(\Pi; x) = \sum_{\sigma \in S_n(\Pi)} F_{\text{Des} \sigma}$.

When is $Q_n(\Pi)$ symmetric?
Let \( x = \{ x_1, x_2, \ldots \} \). The \textit{symmetric functions}, \( \text{Sym}_n \), are \( f(x) \in \mathbb{C}[[x]] \) which are invariant under permutation of variables and homogeneous of degree \( n \). Bases for \( \text{Sym}_n \) are indexed by integer partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \).
Let $\mathbf{x} = \{x_1, x_2, \ldots \}$. The **symmetric functions**, $\text{Sym}_n$, are $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ which are invariant under permutation of variables and homogeneous of degree $n$. Bases for $\text{Sym}_n$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

**Ex.** $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$
Let $x = \{x_1, x_2, \ldots\}$. The **symmetric functions**, $\text{Sym}_n$, are $f(x) \in \mathbb{C}[[x]]$ which are invariant under permutation of variables and homogeneous of degree $n$. Bases for $\text{Sym}_n$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

**Ex.** $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$

**Quasisymmetric functions**, $\text{QSym}_n$, are invariant under order preserving variable substitution and homogeneous of degree $n$. 

---

**Definition:** A **symmetric function** $f(x) \in \mathbb{C}[[x]]$ is a function that is invariant under permutation of variables, i.e., $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$ for any permutation $\sigma$. A symmetric function is homogeneous of degree $n$ if it can be written as a sum of monomials each of degree $n$.

**Example:** Let $x = \{x_1, x_2\}$. Then $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$ is a symmetric function of degree $n$.

**Quasisymmetric functions** are a generalization of symmetric functions, where the order of variables is preserved subject to a given set of relations. They are defined as functions that are invariant under order-preserving variable substitution. A quasisymmetric function is homogeneous of degree $n$ if it can be written as a sum of monomials each of degree $n$ and respects the order of variables.

**Example:** Let $x = \{x_1, x_2\}$. Then $Q_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$ is a quasisymmetric function of degree $n$. The quasisymmetric functions are often used in combinatorics and algebraic combinatorics due to their close connection with other mathematical structures such as Young tableaux and Schur functions.
Let $\mathbf{x} = \{x_1, x_2, \ldots \}$. The **symmetric functions**, $\text{Sym}_n$, are $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ which are invariant under permutation of variables and homogeneous of degree $n$. Bases for $\text{Sym}_n$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

**Ex.** $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$

**Quasisymmetric functions**, $\text{QSym}_n$, are invariant under order preserving variable substitution and homogeneous of degree $n$. Compositions $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash n$ index bases for $\text{QSym}_n$. 
Let \( x = \{x_1, x_2, \ldots \} \). The \textit{symmetric functions}, \( \text{Sym}_n \), are \( f(x) \in \mathbb{C}[[x]] \) which are invariant under permutation of variables and homogeneous of degree \( n \). Bases for \( \text{Sym}_n \) are indexed by integer partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \).

\textbf{Ex.} \( m_{(2,1)} = x_2^2 x_1 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots \)

\textit{Quasisymmetric functions}, \( \text{QSym}_n \), are invariant under order preserving variable substitution and homogeneous of degree \( n \). Compositions \( \alpha = (\alpha_1, \ldots, \alpha_k) \models n \) index bases for \( \text{QSym}_n \).

\textbf{Ex.} \( M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \ldots \)
Let $x = \{x_1, x_2, \ldots \}$. The **symmetric functions**, $\text{Sym}_n$, are $f(x) \in \mathbb{C}[[x]]$ which are invariant under permutation of variables and homogeneous of degree $n$. Bases for $\text{Sym}_n$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

**Ex.** $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$

**Quasisymmetric functions**, $\text{QSym}_n$, are invariant under order preserving variable substitution and homogeneous of degree $n$. Compositions $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash n$ index bases for $\text{QSym}_n$.

**Ex.** $M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \ldots$

Equivalently, bases for $\text{QSym}_n$ are indexed by sets $S \subseteq [n-1]$. 


Let $x = \{x_1, x_2, \ldots \}$. The \textit{symmetric functions}, $\text{Sym}_n$, are $f(x) \in \mathbb{C}[[x]]$ which are invariant under permutation of variables and homogeneous of degree $n$. Bases for $\text{Sym}_n$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

Ex. $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$

\textit{Quasisymmetric functions}, $\text{QSym}_n$, are invariant under order preserving variable substitution and homogeneous of degree $n$. Compositions $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash n$ index bases for $\text{QSym}_n$.

Ex. $M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \ldots$

Equivalently, bases for $\text{QSym}_n$ are indexed by sets $S \subseteq [n - 1]$. The corresponding \textit{fundamental quasisymmetric function} is

$$F_S = \sum x_{i_1} x_{i_2} \ldots x_{i_n}$$

summed over $i_1 \leq i_2 \leq \cdots \leq i_n$ with $i_j < i_{j+1}$ iff $j \in S$. 
Let $x = \{x_1, x_2, \ldots \}$. The **symmetric functions**, $\text{Sym}_n$, are $f(x) \in \mathbb{C}[[x]]$ which are invariant under permutation of variables and homogeneous of degree $n$. Bases for $\text{Sym}_n$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

**Ex.** $m_{(2,1)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \ldots$

**Quasisymmetric functions**, $\text{QSym}_n$, are invariant under order preserving variable substitution and homogeneous of degree $n$. Compositions $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash n$ index bases for $\text{QSym}_n$.

**Ex.** $M_{(1,2)} = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + \ldots$

Equivalently, bases for $\text{QSym}_n$ are indexed by sets $S \subseteq [n-1]$. The corresponding **fundamental quasisymmetric function** is

$$F_S = \sum x_{i_1}x_{i_2}\ldots x_{i_n}$$

summed over $i_1 \leq i_2 \leq \cdots \leq i_n$ with $i_j < i_{j+1}$ iff $j \in S$.

**Ex.** $n = 3$: $F_{\{1\}} = x_1x_2^2 + x_1x_3^2 + \cdots + x_1x_2x_3 + x_1x_2x_4 + \cdots$
Let $x = \{x_1, x_2, \ldots \}$. The **symmetric functions**, $\text{Sym}_n$, are $f(x) \in \mathbb{C}[\lbrack x \rbrack]$ which are invariant under permutation of variables and homogeneous of degree $n$. Bases for $\text{Sym}_n$ are indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

**Ex.** $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$

**Quasisymmetric functions**, $\text{QSym}_n$, are invariant under order preserving variable substitution and homogeneous of degree $n$. Compositions $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash n$ index bases for $\text{QSym}_n$.

**Ex.** $M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \ldots$

Equivalently, bases for $\text{QSym}_n$ are indexed by sets $S \subseteq [n - 1]$.

The corresponding **fundamental quasisymmetric function** is

$$F_S = \sum x_{i_1} x_{i_2} \ldots x_{i_n}$$

summed over $i_1 \leq i_2 \leq \cdots \leq i_n$ with $i_j < i_{j+1}$ iff $j \in S$.

**Ex.** $n = 3$: $F_{\{1\}} = x_1 x_2^2 + x_1 x_3^2 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots$

Given a set of permutations $\Pi$, define

$$Q_n(\Pi) = Q_n(\Pi; x) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\text{Des} \sigma}.$$
Let \( x = \{ x_1, x_2, \ldots \} \). The \textit{symmetric functions}, \( \text{Sym}_n \), are \( f(x) \in \mathbb{C}[[x]] \) which are invariant under permutation of variables and homogeneous of degree \( n \). Bases for \( \text{Sym}_n \) are indexed by integer partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \).

\textbf{Ex.} \( m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots \)

\textit{Quasisymmetric functions}, \( \text{QSym}_n \), are invariant under order preserving variable substitution and homogeneous of degree \( n \). Compositions \( \alpha = (\alpha_1, \ldots, \alpha_k) \vdash n \) index bases for \( \text{QSym}_n \).

\textbf{Ex.} \( M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \ldots \)

Equivalently, bases for \( \text{QSym}_n \) are indexed by sets \( S \subseteq [n-1] \).

The corresponding \textit{fundamental quasisymmetric function} is

\[ F_S = \sum x_{i_1} x_{i_2} \ldots x_{i_n} \]

summed over \( i_1 \leq i_2 \leq \cdots \leq i_n \) with \( i_j < i_{j+1} \) iff \( j \in S \).

\textbf{Ex.} \( n = 3 \): \( F_{\{1\}} = x_1 x_2^2 + x_1 x_3^2 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots \)

Given a set of permutations \( \Pi \), define

\[ Q_n(\Pi) = Q_n(\Pi; x) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\text{Des } \sigma} \]

When is \( Q_n(\Pi) \) symmetric?
All tableaux will be in English notation.
All tableaux will be in English notation. Given a partition $\lambda$, let

$$\text{SYT}(\lambda) = \{ T : T \text{ is a standard Young tableau of shape } \lambda \} ,$$

and

$$\text{SSYT}(\lambda) = \{ T : T \text{ is a semistandard Young tableau of shape } \lambda \} .$$

The Schur function associated to $\lambda$ is

$$s_{\lambda} = \sum_{T \in \text{SSYT}(\lambda)} \prod_{(i,j) \in \lambda} x_{T_{ij}} ,$$

where $T_{ij}$ denotes the entry in the $i$th row and $j$th column of $T$.

Tableau $T \in \text{SYT}(\lambda)$ has descent set $\text{Des}_T = \{ i : i + 1 \text{ is in a lower row than } i \}$.

Theorem (Gessel, 1984)

For any $\lambda \vdash n$

$$s_{\lambda} = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des}_T} .$$

Example:

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

$s_{(3,2)} = F_{\{3\}} + F_{\{2,4\}} + F_{\{2\}} + F_{\{1,4\}} + F_{\{1,3\}}$.
All tableaux will be in English notation. Given a partition \( \lambda \), let

\[
\begin{align*}
\text{SYT}(\lambda) & = \{ T : T \text{ is a standard Young tableau of shape } \lambda \}, \\
\text{SSYT}(\lambda) & = \{ T : T \text{ is a semistandard Young tableau of shape } \lambda \}.
\end{align*}
\]
All tableaux will be in English notation. Given a partition $\lambda$, let

\[
\text{SYT}(\lambda) = \{ T : T \text{ is a standard Young tableau of shape } \lambda \},
\]
\[
\text{SSYT}(\lambda) = \{ T : T \text{ is a semistandard Young tableau of shape } \lambda \}.
\]

The **Schur function** associated to $\lambda$ is

\[
s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \prod_{(i,j) \in \lambda} x_{T_{i,j}}.
\]
All tableaux will be in English notation. Given a partition $\lambda$, let

$$\text{SYT}(\lambda) = \{ T : T \text{ is a standard Young tableau of shape } \lambda \},$$

$$\text{SSYT}(\lambda) = \{ T : T \text{ is a semistandard Young tableau of shape } \lambda \}.$$  

The *Schur function* associated to $\lambda$ is

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \prod_{(i,j) \in \lambda} x_{T_{i,j}}.$$  

Tableau $T \in \text{SYT}(T)$ has *descent set*

$$\text{Des } T = \{ i : i + 1 \text{ is in a lower row than } i \}.$$
All tableaux will be in English notation. Given a partition $\lambda$, let

\[ \text{SYT}(\lambda) = \{ T : T \text{ is a standard Young tableau of shape } \lambda \} , \]
\[ \text{SSYT}(\lambda) = \{ T : T \text{ is a semistandard Young tableau of shape } \lambda \} . \]

The \textit{Schur function} associated to $\lambda$ is

\[ s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \prod_{(i,j) \in \lambda} x_{T_{i,j}} . \]

Tableau $T \in \text{SYT}(\lambda)$ has \textit{descent set}

\[ \text{Des } T = \{ i : i + 1 \text{ is in a lower row than } i \} . \]

\textbf{Theorem (Gessel, 1984)}

\textit{For any } $\lambda \vdash n$

\[ s_\lambda = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des } T} . \]
All tableaux will be in English notation. Given a partition $\lambda$, let

$\text{SYT}(\lambda) = \{ T : T \text{ is a standard Young tableau of shape } \lambda \}$,

$\text{SSYT}(\lambda) = \{ T : T \text{ is a semistandard Young tableau of shape } \lambda \}$.

The *Schur function* associated to $\lambda$ is

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \prod_{(i,j) \in \lambda} x_{T_{i,j}}.$$ 

Tableau $T \in \text{SYT}(\lambda)$ has *descent set*

$$\text{Des } T = \{ i : i + 1 \text{ is in a lower row than } i \}.$$ 

**Theorem (Gessel, 1984)**

*For any $\lambda \vdash n$*

$$s_\lambda = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des } T}.$$ 

**Ex.** $T :$

\[
\begin{array}{ccc}
\text{1} & \text{2} & \text{3} \\
\text{4} & \text{5} & \\
\end{array}
\quad
\begin{array}{cccc}
\text{1} & \text{2} & \text{4} & \text{5} \\
\text{3} & \text{4} & \text{5} & \\
\text{1} & \text{3} & \text{4} & \\
\end{array}
\quad
\begin{array}{cccc}
\text{1} & \text{3} & \text{5} & \\
\text{2} & \text{5} & \\
\text{2} & \text{5} & \\
\end{array}
\]
All tableaux will be in English notation. Given a partition $\lambda$, let

$$\text{SYT}(\lambda) = \{ T : T \text{ is a standard Young tableau of shape } \lambda \}$$

$$\text{SSYT}(\lambda) = \{ T : T \text{ is a semistandard Young tableau of shape } \lambda \}$$.

The *Schur function* associated to $\lambda$ is

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \prod_{(i,j) \in \lambda} x_{T_{i,j}}.$$ 

Tableau $T \in \text{SYT}(\lambda)$ has *descent set*

$$\text{Des } T = \{ i : i + 1 \text{ is in a lower row than } i \}.$$ 

**Theorem (Gessel, 1984)**

For any $\lambda \vdash n$

$$s_\lambda = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des } T}.$$ 

**Ex.** $T$:

<table>
<thead>
<tr>
<th>1 2 3</th>
<th>1 2 4</th>
<th>1 2 5</th>
<th>1 3 4</th>
<th>1 3 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 5</td>
<td>4 3</td>
<td>5 3</td>
<td>5 2</td>
<td>5 2</td>
</tr>
</tbody>
</table>

$$s_{(3,2)} = F_{\{3\}} + F_{\{2,4\}} + F_{\{2\}} + F_{\{1,4\}} + F_{\{1,3\}}.$$
Outline

Patterns and quasisymmetric functions

Main result

Questions
Call \( f(x) \in \text{Sym}_n \) \textit{Schur nonnegative} if \( f(x) = \sum_{\lambda} c_{\lambda} s_{\lambda} \) with \( c_{\lambda} \geq 0 \) for all \( \lambda \).
Call $f(x) \in \text{Sym}_n$ \textit{Schur nonnegative} if $f(x) = \sum_{\lambda} c_{\lambda} s_{\lambda}$ with $c_{\lambda} \geq 0$ for all $\lambda$.

**Theorem (S.)**

\textit{Suppose $\{123, 321\} \not\subseteq \Pi \subseteq S_3$.}
Call $f(x) \in \text{Sym}_n$ **Schur nonnegative** if $f(x) = \sum_{\lambda} c_{\lambda} s_{\lambda}$ with $c_{\lambda} \geq 0$ for all $\lambda$.

**Theorem (S.)**

*Suppose* $\{123, 321\} \not\subseteq \Pi \subseteq S_3$. TFAE

1. $Q_n(\Pi)$ *is symmetric for all* $n$.  

In all cases, $\lambda$ runs over partitions of $n$, $f_{\lambda} = |\text{SYT}(\lambda)|$, and $c(\lambda)$ and $r(\lambda)$ are the number of columns and rows of $\lambda$.  


Call $f(x) \in \text{Sym}_n$ **Schur nonnegative** if $f(x) = \sum_{\lambda} c_{\lambda} s_{\lambda}$ with $c_{\lambda} \geq 0$ for all $\lambda$.

**Theorem (S.)**

Suppose $\{123, 321\} \not\subseteq \Pi \subseteq S_3$. TFAE

1. $Q_n(\Pi)$ is symmetric for all $n$.
2. $Q_n(\Pi)$ is Schur nonnegative for all $n$.  

In all cases, $\lambda$ runs over partitions of $n$, $f_{\lambda} = |\text{SYT}(\lambda)|$, and $c(\lambda)$ and $r(\lambda)$ are the number of columns and rows of $\lambda$. 
Call \( f(x) \in \text{Sym}_n \) **Schur nonnegative** if \( f(x) = \sum_{\lambda} c_{\lambda} s_{\lambda} \) with \( c_{\lambda} \geq 0 \) for all \( \lambda \).

**Theorem (S.)**

*Suppose \( \{123, 321\} \not\subseteq \Pi \subseteq \mathfrak{S}_3 \). TFAE*

1. \( Q_n(\Pi) \) is symmetric for all \( n \).
2. \( Q_n(\Pi) \) is Schur nonnegative for all \( n \).
3. \( \Pi \) is an entry in the following table.

<table>
<thead>
<tr>
<th>( \Pi )</th>
<th>( Q_n(\Pi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>( \sum_{\lambda} f^\lambda s_{\lambda} )</td>
</tr>
<tr>
<td>( {123} )</td>
<td>( \sum_{c(\lambda) \leq 2} f^\lambda s_{\lambda} )</td>
</tr>
<tr>
<td>( {321} )</td>
<td>( \sum_{r(\lambda) \leq 2} f^\lambda s_{\lambda} )</td>
</tr>
<tr>
<td>( {132, 213}; {132, 213}; {132, 312}; {132, 312} )</td>
<td>( \sum_{\lambda \text{ a hook}} s_{\lambda} )</td>
</tr>
<tr>
<td>( {123, 132, 312}; {123, 213, 231}; {123, 231, 312} )</td>
<td>( s_{1n} + s_{2,1n-2} )</td>
</tr>
<tr>
<td>( {132, 213, 321}; {132, 312, 321}; {213, 231, 321} )</td>
<td>( s_n + s_{n-1,1} )</td>
</tr>
<tr>
<td>( {132, 213, 231, 312} )</td>
<td>( s_n + s_{1n} ).</td>
</tr>
</tbody>
</table>

*In all cases, \( \lambda \) runs over partitions of \( n \), \( f^\lambda = |\text{SYT}(\lambda)| \), and \( c(\lambda) \) and \( r(\lambda) \) are the number of columns and rows of \( \lambda \).*
Proposition

We have

\[ Q_n(132, 213) = \sum_{\lambda \text{ a hook}} s_\lambda. \]

Proof (Sketch).

A permutation \( \sigma \in S_n \) is skew layered if it is of the form

\[ \sigma = m, m+1, \ldots, n, l, l+1, \ldots, m-1, \ldots, 2, k-1, \ldots, 1. \]

\[ S_n(132, 213) = \{ \sigma \in S_n : \sigma \text{ is skew layered} \}. \]

\[ \therefore \] \[ Q_n(132, 213) = \sum_{S \subseteq [n-1]} F_S. \]

By Gessel's Theorem, if \( a+b = n \), \( s_a, 1 s_b = \sum_{S \subseteq [n-1], |S| = b} F_S. \)

Combining the two previous equations completes the proof.
Proposition

We have

\[ Q_n(132, 213) = \sum_{\lambda \text{ a hook}} s_\lambda. \]

Proof (Sketch).

A permutation \( \sigma \in S_n \) is *skew layered* if it is of the form

\[ \sigma = m, m + 1, \ldots, n, l, l + 1, \ldots, m - 1, \ldots, 1, 2, \ldots, k - 1. \]
Proposition

We have

\[ Q_n(132, 213) = \sum_{\lambda \text{ a hook}} s_\lambda. \]

Proof (Sketch).

A permutation \( \sigma \in \mathfrak{S}_n \) is **skew layered** if it is of the form

\[ \sigma = m, m+1, \ldots, n, l, l+1, \ldots, m-1, \ldots, 1, 2, \ldots, k-1. \]

\[ \mathfrak{S}_n(132, 213) = \{ \sigma \in \mathfrak{S}_n : \sigma \text{ is skew layered} \}. \]
Proposition

We have

\[ Q_n(132, 213) = \sum_{\lambda \text{ a hook}} s_{\lambda}. \]

Proof (Sketch).

A permutation \( \sigma \in \mathfrak{S}_n \) is \textit{skew layered} if it is of the form

\[ \sigma = m, m + 1, \ldots, n, l, l + 1, \ldots, m - 1, \ldots, 1, 2, \ldots, k - 1. \]

\[ \mathfrak{S}_n(132, 213) = \{ \sigma \in \mathfrak{S}_n : \sigma \text{ is skew layered} \}. \]

\[ \therefore \quad Q_n(132, 213) = \sum_{S \subseteq [n-1]} F_S. \]
Proposition

We have

\[ Q_n(132, 213) = \sum_{\lambda \text{ a hook}} s_{\lambda}. \]

Proof (Sketch).

A permutation \( \sigma \in S_n \) is \textit{skew layered} if it is of the form

\[ \sigma = m, m+1, \ldots, n, l, l+1, \ldots, m-1, \ldots, 1, 2, \ldots, k-1. \]

\[ S_n(132, 213) = \{ \sigma \in S_n : \sigma \text{ is skew layered} \}. \]

\[ \therefore Q_n(132, 213) = \sum_{S \subseteq [n-1]} F_S. \]

By Gessel’s Theorem, if \( a + b = n \),

\[ s_{a,1b} = \sum_{S \subseteq [n-1], |S|=b} F_S. \]
Proposition

We have

\[ Q_n(132, 213) = \sum_{\lambda \text{ a hook}} s_\lambda. \]

Proof (Sketch).

A permutation \( \sigma \in \mathcal{S}_n \) is *skew layered* if it is of the form

\[ \sigma = m, m + 1, \ldots, n, l, l + 1, \ldots, m - 1, \ldots, 1, 2, \ldots, k - 1. \]

\[ \mathcal{S}_n(132, 213) = \{ \sigma \in \mathcal{S}_n : \sigma \text{ is skew layered} \}. \]

\[ \therefore Q_n(132, 213) = \sum_{S \subseteq [n-1]} F_S. \]

By Gessel’s Theorem, if \( a + b = n \),

\[ s_{a,1}^b = \sum_{S \subseteq [n-1], |S| = b} F_S. \]

Combining the two previous equations completes the proof.
Outline

Patterns and quasisymmetric functions

Main result

Questions
1. Can one give a nice characterization of the $\Pi$ which yield symmetric $Q_n(\Pi)$?
1. Can one give a nice characterization of the $\Pi$ which yield symmetric $Q_n(\Pi)$?
2. Why are all the symmetric $Q_n(\Pi)$ in the main theorem also Schur nonnegative?
1. Can one give a nice characterization of the $\Pi$ which yield symmetric $Q_n(\Pi)$?

2. Why are all the symmetric $Q_n(\Pi)$ in the main theorem also Schur nonnegative?

3. In particular, for these $\Pi$ is there a natural way to associate an $\mathfrak{S}_n$ module whose image under the characteristic map (of its character) is $Q_n(\Pi)$?
1. Can one give a nice characterization of the $\Pi$ which yield symmetric $Q_n(\Pi)$?

2. Why are all the symmetric $Q_n(\Pi)$ in the main theorem also Schur nonnegative?

3. In particular, for these $\Pi$ is there a natural way to associate an $\mathcal{S}_n$ module whose image under the characteristic map (of its character) is $Q_n(\Pi)$? For example, if $\Pi = \emptyset$ then $\mathcal{S}_n(\Pi) = \mathcal{S}_n$ which is a module for the regular representation.
1. Can one give a nice characterization of the $\Pi$ which yield symmetric $Q_n(\Pi)$?

2. Why are all the symmetric $Q_n(\Pi)$ in the main theorem also Schur nonnegative?

3. In particular, for these $\Pi$ is there a natural way to associate an $\mathfrak{S}_n$ module whose image under the characteristic map (of its character) is $Q_n(\Pi)$? For example, if $\Pi = \emptyset$ then $\mathfrak{S}_n(\Pi) = \mathfrak{S}_n$ which is a module for the regular representation. Applying the characteristic map gives

$$\sum_{\lambda \vdash n} f^\lambda s_\lambda = Q_n(\Pi).$$
THANKS FOR LISTENING!