Möbius Functions of Posets II: Möbius Inversion

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The Incidence Algebra

The Möbius Function

The Möbius Inversion Theorem

Outline

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Note. Often $\cdot$ and $\bullet$ are suppressed since context makes it clear which multiplication is meant.

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Outline

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The Möbius Function

The Möbius Inversion Theorem

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Note. If P has a zero then we write

$$\mu(y) = \mu(\hat{0}, y).$$

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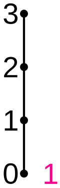
$C_3=$ 

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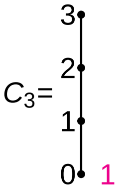

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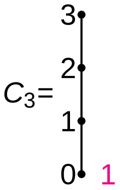

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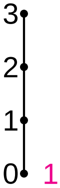
Example: The Chain.

$C_3 =$



$$\mu(0) = \mu(\hat{0}) = 1,$$
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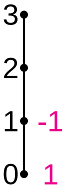

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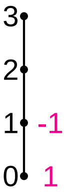
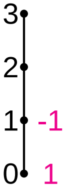


$$\mu(0) = \mu(\hat{0}) = 1,$$
$$\mu(1) = -\mu(0) = -1,$$
$$\mu(2)$$

$$\mu(y) = \begin{cases} 1 & \text{if } y = \hat{0}, \\ -\sum_{z<y} \mu(z) & \text{if } y > \hat{0}. \end{cases}$$

Example: The Chain.

$C_3 =$



$$\mu(0) = \mu(\hat{0}) = 1,$$
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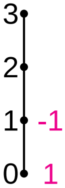
$$\mu(y) = \begin{cases} 1 & \text{if } y = \hat{0}, \\ -\sum_{z<y} \mu(z) & \text{if } y > \hat{0}. \end{cases}$$

Example: The Chain.

$C_3 =$ 

$$\mu(0) = \mu(\hat{0}) = 1,$$
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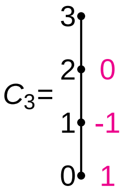
Example: The Chain.



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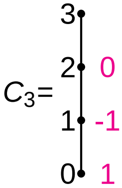

$$\begin{aligned} &\mu(0) = \mu(\hat{0}) = 1, \\ &\mu(1) = -\mu(0) = -1, \\ &\mu(2) = -(\mu(0) + \mu(1)) = -(1 - 1) = 0, \\ &\mu(3) \end{aligned}$$

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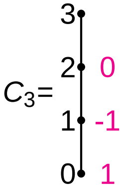
Example: The Chain.



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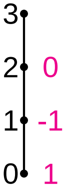

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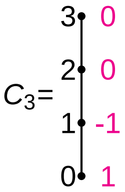
Example: The Chain.



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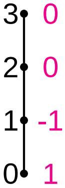

$$\begin{aligned}
\mu(0) &= \mu(\hat{0}) = 1, \\
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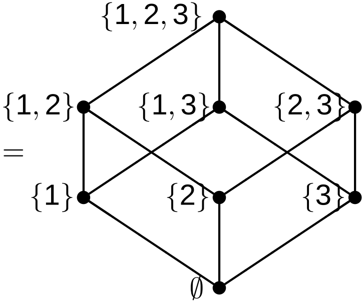
\mu(3) &= -(\mu(0) + \mu(1) + \mu(2)) = -(1 - 1 + 0) = 0,
\end{aligned}$$

Proposition

In C_n we have $\mu(i, j) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \lhd j, \\ 0 & \text{else.} \end{cases}$ $\square$

Example: The Boolean Algebra.

$B_3 =$



Example: The Boolean Algebra.

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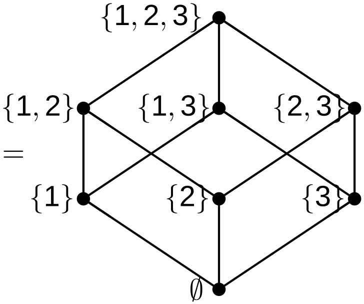
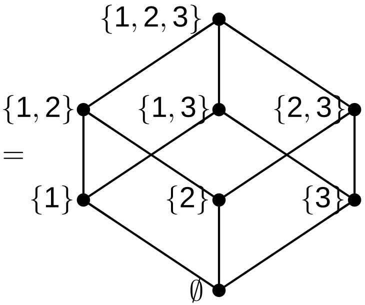


$\mu(\emptyset)$

Example: The Boolean Algebra.

$B_3 =$



$$\mu(\emptyset) = \mu(\hat{0}) = 1,$$

Example: The Boolean Algebra.

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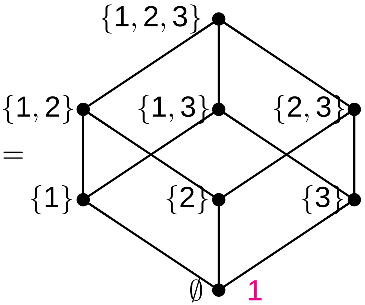

$\mu(\emptyset) = \mu(\hat{0}) = 1,$

Example: The Boolean Algebra.

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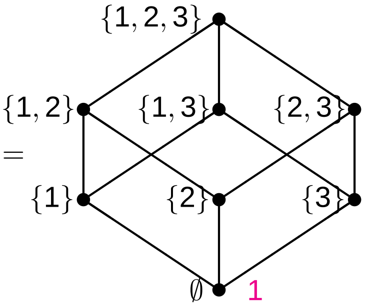
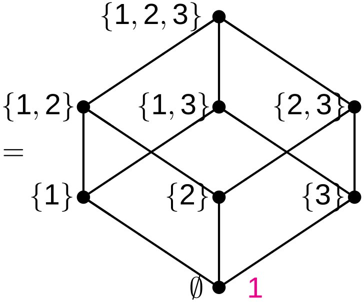


$\mu(\emptyset) = \mu(\hat{0}) = 1,$
$\mu(\{1\})$

Example: The Boolean Algebra.

$B_3 =$ 

$$\mu(\emptyset) = \mu(\hat{0}) = 1,$$
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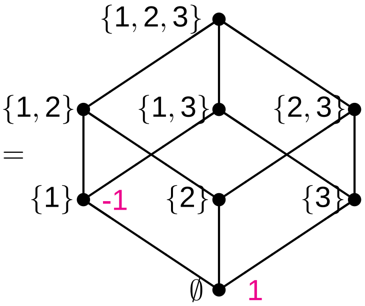

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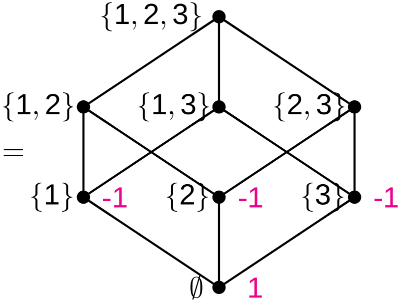

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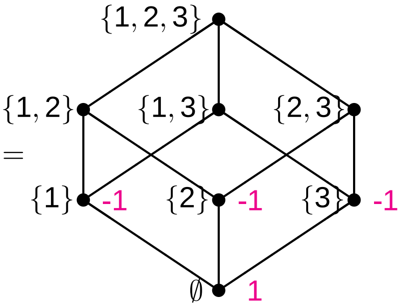
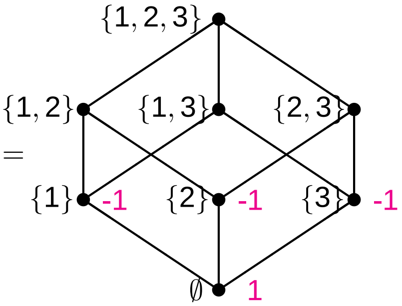


$$\mu(\emptyset) = \mu(\hat{0}) = 1,$$
$$\mu(\{1\}) = -\mu(\emptyset) = -1,$$
$$\mu(\{1, 2\})$$

Example: The Boolean Algebra.

$B_3 =$



$$\mu(\emptyset) = \mu(\hat{0}) = 1,$$
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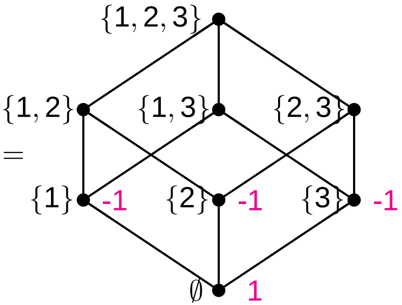

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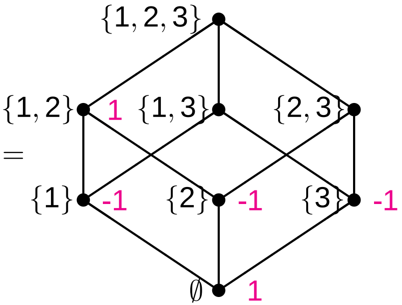
Example: The Boolean Algebra.

$B_3 =$



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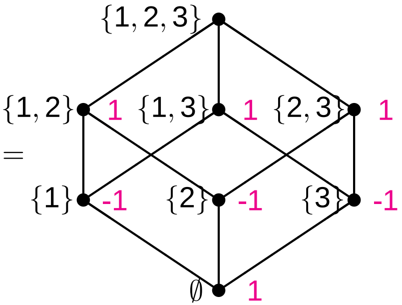
$B_3 =$ 

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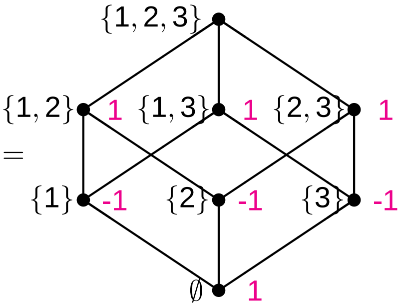

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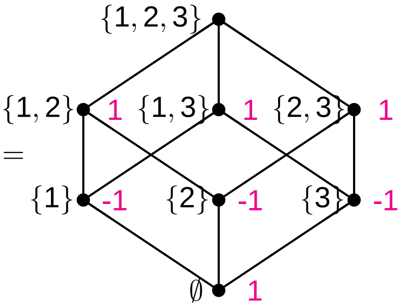
$$\mu(\{1,2,3\})$$

Example: The Boolean Algebra.

$B_3 =$ 

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$$\mu(\{1,2,3\}) = -(1-1-1-1+1+1+1) = -1$$

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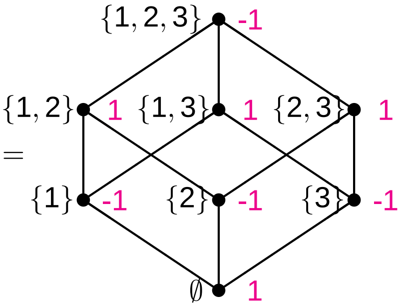

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Example: The Boolean Algebra.

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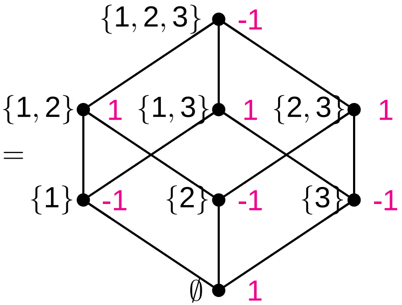

$$
\begin{aligned}
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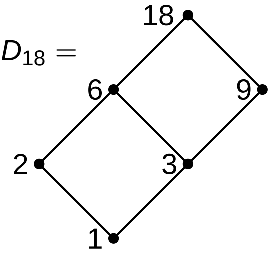
&\mu(\{1,2\}) = -(\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1,\\
&\mu(\{1,2,3\}) = -(1 - 1 - 1 - 1 + 1 + 1 + 1) = -1
\end{aligned}
$$

Conjecture

In B_n we have $\mu(S) = (-1)^{|S|}$.

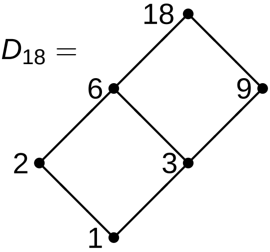
Example: The Divisor Lattice.

$D_{18} =$



Example: The Divisor Lattice.

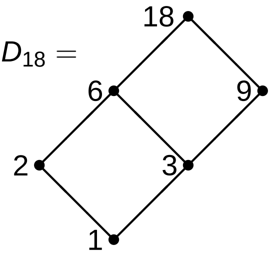
$D_{18} =$



$\mu(1)$

Example: The Divisor Lattice.

$D_{18} =$



$\mu(1) = \mu(\hat{0}) = 1,$

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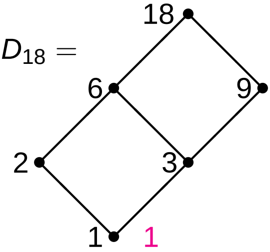

$\mu(1) = \mu(\hat{0}) = 1,$

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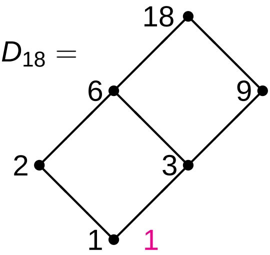

$\mu(1) = \mu(\hat{0}) = 1,$
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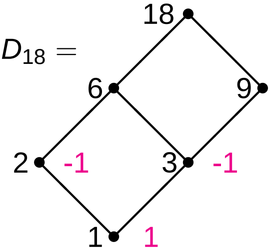

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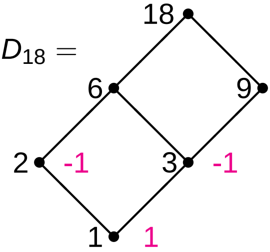

$\mu(1) = \mu(\hat{0}) = 1,$
$\mu(2) = \mu(3) = -1,$
$\mu(6)$

Example: The Divisor Lattice.

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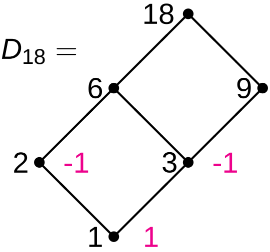

$$\mu(1) = \mu(\hat{0}) = 1,$$
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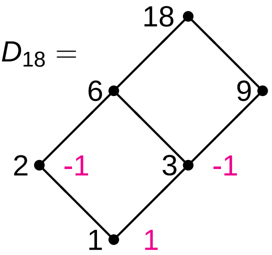

$$\mu(1) = \mu(\hat{0}) = 1,$$
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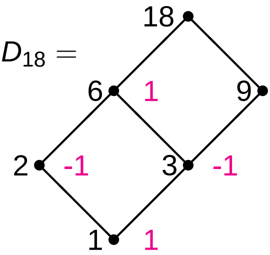

$$\mu(1) = \mu(\hat{0}) = 1,$$
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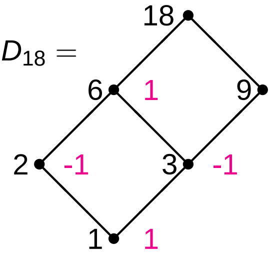

$\mu(1) = \mu(\hat{0}) = 1,$
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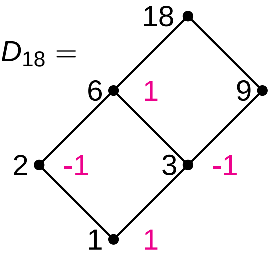
$\mu(9)$

Example: The Divisor Lattice.

$D_{18} =$



$$\mu(1) = \mu(\hat{0}) = 1,$$
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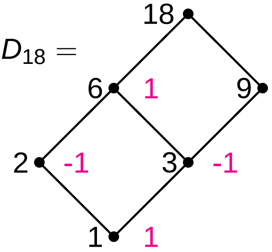
$$\mu(9) = -(\mu(1) + \mu(3))$$

Example: The Divisor Lattice.

$D_{18} =$



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Example: The Divisor Lattice.

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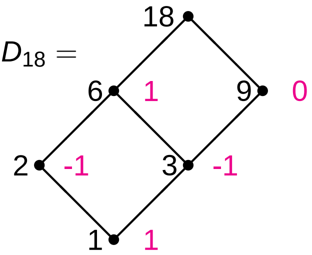

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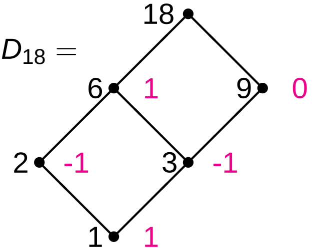

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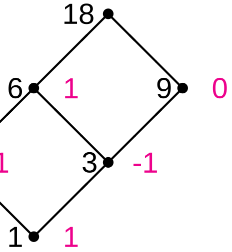

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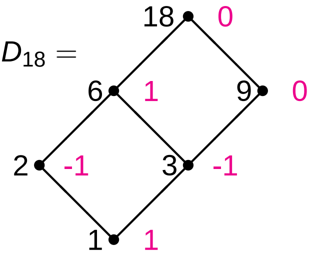

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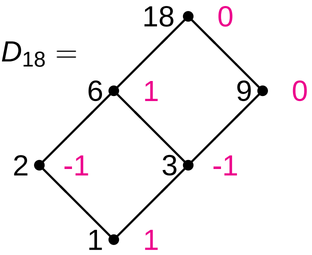

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Conjecture

If $d \in D_n$ has prime factorization $d = p_1^{m_1} \cdots p_k^{m_k}$ then

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Outline

Theorem (Möbius Inversion Theorem or MIT, Weisner (1935))

Consider a finite poset P and two functions $f : P \to \mathbb{R}$ and $g : P \to \mathbb{R}$. Then the following are equivalent statements.

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Example: Theory of Finite Differences.

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For $g : \mathbb{Z}_{\ge 0} \to \mathbb{R}$: $\Delta g(n) = g(n) - g(n-1)$, $\quad Sg(n) = \sum_{i=0}^{n} g(i)$.

Theorem (FTDC)

If $g : \mathbb{Z}_{\ge 0} \to \mathbb{R}$ *then:* $\qquad \Delta Sg(n) = g(n)$.

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Let U be a finite set and $U_1, \ldots, U_n \subseteq U$.

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$$|U-\bigcup_{i=1}^{n}U_i|=g(\emptyset)$$

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Let $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$ satisfy $f(n) = \sum_{d|n} g(d)$ for all $n \in \mathbb{Z}_{>0}$. Then

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