Möbius Functions of Posets II: Möbius Inversion

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The Incidence Algebra

The Möbius Function

The Möbius Inversion Theorem
The Incidence Algebra

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The Möbius Inversion Theorem
The *incidence algebra* of a finite poset $P$ is the set

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**Example.** $I(P)$ has Kronecker’s delta: $\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$
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since $\alpha(x, z) \neq 0$ implies $x \leq z$ and $\beta(z, y) \neq 0$ implies $z \leq y$. 
An *algebra* over a field $F$ is a set $A$ together with operations of sum ($+$), product ($\bullet$), and scalar multiplication ($\cdot$) such that

1. $(A, +, \bullet)$ is a ring,
2. $(A, +, \cdot)$ is a vector space over $F$,
3. $k \cdot (a \bullet b) = (k \cdot a) \bullet b = a \bullet (k \cdot b)$ for all $k \in F$, $a, b \in A$.

Example. The matrix algebra over $\mathbb{R}$ is $\text{Mat}_n(\mathbb{R}) = \text{all } n \times n \text{ matrices with entries in } \mathbb{R}$.

Example. The Boolean algebra is an algebra over $\mathbb{F}_2$ where, for all $S, T \in B_n$:

1. $S + T = (S \cup T) - (S \cap T)$,
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Let $L : x_1, \ldots, x_n$ be a list of the elements of $P$. An $L \times L$ matrix has rows and columns indexed by $L$. 

The matrix algebra of $P$ is $M(P) = \{ M \in \text{Mat}_n(\mathbb{R}) | M \text{ is } L \times L \text{ and } M_{x,y} = 0 \text{ if } x \not\leq y \}$. 

Note that $M(P)$ is a subalgebra of $\text{Mat}_n(\mathbb{R})$.

Example. For $B^2_2$, let $L : \emptyset, \{1\}, \{2\}, \{1,2\}$. Then a typical element of $M(B^2_2)$ is 

$$M = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & 0 & \bullet & \bullet \\ \bullet & \bullet & 0 & \bullet \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}$$

where the $\bullet$'s can be replace by any complex numbers.
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\{1\} & 0 & \diamondsuit & 0 \\
\{2\} & 0 & 0 & \diamondsuit \\
\{1,2\} & \diamondsuit & \diamondsuit & \diamondsuit
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$$M = \begin{bmatrix}
\emptyset & \{1\} & \{2\} & \{1,2\} \\
\{1\} & 0 & \diamondsuit & 0 & \diamondsuit \\
\{2\} & 0 & 0 & \diamondsuit & \diamondsuit \\
\{1,2\} & 0 & 0 & 0 & \diamondsuit \\
\end{bmatrix}$$

where the ♦’s can be replaced by any complex numbers.
Let $L : x_1, \ldots, x_n$ be a list of the elements of $P$. An $L \times L$ matrix has rows and columns indexed by $L$. The matrix algebra of $P$ is

$$M(P) = \{ M \in \text{Mat}_n(\mathbb{R}) \mid M \text{ is } L \times L \text{ and } M_{x,y} = 0 \text{ if } x \nleq y. \}$$

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An *isomorphism* of algebras $A$ and $B$ is a bijection $f : A \rightarrow B$ such that for all $a, b \in A$ and $k \in F$,

$$f(a + b) = f(a) + f(b), \quad f(a \cdot b) = f(a) \cdot f(b), \quad f(k \cdot a) = k \cdot f(a).$$
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Given any $\alpha \in \mathcal{I}(P)$ we let $M^\alpha$ be the matrix with entries

$$M^\alpha_{x,y} = \alpha(x, y).$$
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**Theorem**

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**Theorem**

*The map $\alpha \mapsto M^\alpha$ is an algebra isomorphism $I(P) \rightarrow M(P)$.***

**Proof** that product is preserved.

We wish to show $M^\alpha \ast M^\beta = M^\alpha M^\beta$.

But given $x, y \in P$:

$$M^\alpha \ast M^\beta x, y = (\alpha \ast \beta)(x, y) = \sum z \alpha(x, z) \beta(z, y) = (M^\alpha M^\beta)x, y.$$
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**Theorem**

The map $\alpha \mapsto M^\alpha$ is an algebra isomorphism $I(P) \to M(P)$.

**Proof that product is preserved.** We wish to show $M^{\alpha \cdot \beta} = M^\alpha M^\beta$. 
An \textit{isomorphism} of algebras $A$ and $B$ is a bijection $f : A \rightarrow B$ such that for all $a, b \in A$ and $k \in F$,
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M^\alpha_{x,y} = \alpha(x, y).
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\textbf{Theorem}

\textit{The map }$\alpha \mapsto M^\alpha$ \textit{is an algebra isomorphism }$I(P) \rightarrow M(P)$.

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Given any $\alpha \in l(P)$ we let $M^\alpha$ be the matrix with entries

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**Theorem**

*The map $\alpha \mapsto M^\alpha$ is an algebra isomorphism $l(P) \to M(P)$.*

**Proof that product is preserved.** We wish to show $M^{\alpha \ast \beta} = M^\alpha M^\beta$. But given $x, y \in P$:

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**Theorem**

The map $\alpha \mapsto M^\alpha$ is an algebra isomorphism $I(P) \rightarrow M(P)$.

**Proof that product is preserved.** We wish to show $M^{\alpha \ast \beta} = M^\alpha M^\beta$. But given $x, y \in P$:

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**Proposition**

*If $\alpha \in l(P)$ then $\alpha^{-1}$ exists if and only if $\alpha(x, x) \neq 0$ for all $x \in P$.***
An isomorphism of algebras $A$ and $B$ is a bijection $f : A \to B$ such that for all $a, b \in A$ and $k \in F$,
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**Theorem**
The map $\alpha \mapsto M^\alpha$ is an algebra isomorphism $I(P) \to M(P)$.

**Proof that product is preserved.** We wish to show $M^{\alpha \ast \beta} = M^\alpha M^\beta$. But given $x, y \in P$:
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M^{\alpha \ast \beta}_{x,y} = (\alpha \ast \beta)(x, y) = \sum_z \alpha(x, z)\beta(z, y) = (M^\alpha M^\beta)_{x,y}.
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**Proposition**
If $\alpha \in I(P)$ then $\alpha^{-1}$ exists if and only if $\alpha(x, x) \neq 0$ for all $x \in P$.

**Proof.** By the previous theorem
\[
\exists \alpha^{-1}
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**Theorem**

The map $\alpha \mapsto M^\alpha$ is an algebra isomorphism $I(P) \to M(P)$.

**Proof that product is preserved.** We wish to show

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But given $x, y \in P$:

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Outline

The Incidence Algebra

The Möbius Function

The Möbius Inversion Theorem
The *zeta function* of $P$ is $\zeta \in \mathcal{I}(P)$ defined by

$$\zeta(x, y) = \begin{cases} 
1 & \text{if } x \leq y, \\
0 & \text{if } x \not\leq y.
\end{cases}$$
The zeta function of $P$ is $\zeta \in I(P)$ defined by

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x \not\leq y. \end{cases}$$

The Möbius function of $P$ is $\mu = \zeta^{-1}$. 
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So if $x = y$ then $\mu(x, x) = 1$; if $x < y$ then $\sum_{z \in [x, y]} \mu(x, z) = 0$. 

Note. If $P$ has a zero then we write $\mu(y) = \mu(\hat{0}, y)$. 


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$$
\mu(x, y) = \begin{cases} 
1 & \text{if } x = y, \\
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\end{cases}
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$$\delta(x, y) = (\mu * \zeta)(x, y) = \sum_{z \in [x, y]} \mu(x, z) \zeta(z, y) = \sum_{z \in [x, y]} \mu(x, z).$$

So if $x = y$ then $\mu(x, x) = 1$; if $x < y$ then $\sum_{z \in [x, y]} \mu(x, z) = 0$. Equivalently

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{z \in [x, y]} \mu(x, z) & \text{if } x < y. \end{cases}$$

Note. If $P$ has a zero then we write

$$\mu(y) = \mu(\hat{0}, y).$$
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}. 
\end{cases} \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array} \]
\( \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \)

Example: The Chain.

\[ C_3 = \]

\[ \mu(0) \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \]

\[ \mu(0) = \mu(\hat{0}) \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}. 
\end{cases} \]

**Example: The Chain.**

\[ C_3 = \begin{array}{c}
\mu(0) = \mu(\hat{0}) = 1,
\end{array} \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \begin{array}{c}
3 \\
2 \\
1 \\
0 \\
1
\end{array} \]

\[ \mu(0) = \mu(\hat{0}) = 1, \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \begin{array}{c}
3 \\
\text{C3=} \\
2 \\
1 \\
0
\end{array} \]

\[ \mu(0) = \mu(\hat{0}) = 1, \]

\[ \mu(1) \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \]

\[ \mu(0) = \mu(\hat{0}) = 1, \]
\[ \mu(1) = -\mu(0) \]
\[\mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
-\sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases}\]

Example: The Chain.

\[C_3 = \begin{array}{c}
& & 3 \\
& 2 & \\
& 1 & \\
0 & & 1
\end{array}\]

\[\mu(0) = \mu(\hat{0}) = 1,\]
\[\mu(1) = -\mu(0) = -1,\]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z<y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \begin{pmatrix}
3 \\
2 \\
1 \\
0
\end{pmatrix} \]

\( \mu(0) = \mu(\hat{0}) = 1, \)

\( \mu(1) = -\mu(0) = -1, \)
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

**Example: The Chain.**

\[ C_3 = \]

\[ \mu(0) = \mu(\hat{0}) = 1, \]
\[ \mu(1) = -\mu(0) = -1, \]
\[ \mu(2) \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
-\sum_{z<y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array} \]

\[ \mu(0) = \mu(\hat{0}) = 1, \]
\[ \mu(1) = -\mu(0) = -1, \]
\[ \mu(2) = -(\mu(0) + \mu(1)) \]
$$\mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
-\sum_{z<y} \mu(z) & \text{if } y > \hat{0}.
\end{cases}$$

Example: The Chain.

$$C_3 = \begin{array}{c}
3 \\
2 \\
1 \\
0 \\
\end{array}$$

$$\mu(0) = \mu(\hat{0}) = 1,$$
$$\mu(1) = -\mu(0) = -1,$$
$$\mu(2) = -(\mu(0) + \mu(1)) = -(1 - 1) = 0,$$
\[
\mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases}
\]

Example: The Chain.

\[
C_3 = \begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array}
\]

\[
\mu(0) = \mu(\hat{0}) = 1, \\
\mu(1) = -\mu(0) = -1, \\
\mu(2) = -(\mu(0) + \mu(1)) = -(1 - 1) = 0,
\]
\[
\mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
-\sum_{z<y} \mu(z) & \text{if } y > \hat{0}.
\end{cases}
\]

Example: The Chain.

\[
C_3 = \begin{bmatrix} 
0 \\
-1 \\
1 \\
1 
\end{bmatrix}
\]

\[
\mu(0) = \mu(\hat{0}) = 1, \\
\mu(1) = -\mu(0) = -1, \\
\mu(2) = -(\mu(0) + \mu(1)) = -(1 - 1) = 0, \\
\mu(3)
\]
\[
\mu(y) = \begin{cases}
1 & \text{if } y = \hat{0}, \\
- \sum_{z < y} \mu(z) & \text{if } y > \hat{0}.
\end{cases}
\]

Example: The Chain.

\[
C_3 = \begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array} \begin{array}{c}
0 \\
-1 \\
1
\end{array}
\]

\[
\mu(0) = \mu(\hat{0}) = 1,
\mu(1) = -\mu(0) = -1,
\mu(2) = -(\mu(0) + \mu(1)) = -(1 - 1) = 0,
\mu(3) = -(\mu(0) + \mu(1) + \mu(2))
\]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
-\sum_{z<y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

**Example: The Chain.**

\[ C_3 = \begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array} \]

- \( \mu(0) = \mu(\hat{0}) = 1, \)
- \( \mu(1) = -\mu(0) = -1, \)
- \( \mu(2) = -(\mu(0) + \mu(1)) = -(1 - 1) = 0, \)
- \( \mu(3) = -(\mu(0) + \mu(1) + \mu(2)) = -(1 - 1 + 0) = 0, \)
\[ \mu(y) = \begin{cases} 1 & \text{if } y = \hat{0}, \\ -\sum_{z<y} \mu(z) & \text{if } y > \hat{0}. \end{cases} \]

Example: The Chain.

\[ C_3 = \begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array} \]

\[ \begin{align*}
\mu(0) &= \mu(\hat{0}) = 1, \\
\mu(1) &= -\mu(0) = -1, \\
\mu(2) &= -(\mu(0) + \mu(1)) = -(1 - 1) = 0, \\
\mu(3) &= -(\mu(0) + \mu(1) + \mu(2)) = -(1 - 1 + 0) = 0,
\end{align*} \]
\[ \mu(y) = \begin{cases} 
1 & \text{if } y = \hat{0}, \\
- \sum_{z<y} \mu(z) & \text{if } y > \hat{0}.
\end{cases} \]

Example: The Chain.

\[ C_3 = \begin{array}{cccc}
0 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
3 & 0 & -1 & 0
\end{array} \]

\[ \mu(0) = \mu(\hat{0}) = 1, \]
\[ \mu(1) = -\mu(0) = -1, \]
\[ \mu(2) = -(\mu(0) + \mu(1)) = -(1 - 1) = 0, \]
\[ \mu(3) = -(\mu(0) + \mu(1) + \mu(2)) = -(1 - 1 + 0) = 0, \]

Proposition

In \( C_n \) we have \( \mu(i, j) = \begin{cases} 
1 & \text{if } i = j, \\
-1 & \text{if } i < j, \\
0 & \text{else}.
\end{cases} \)
Example: The Boolean Algebra.

\[ B_3 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \} \]
Example: The Boolean Algebra.

$B_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$\mu(\emptyset) = 1$
Example: The Boolean Algebra.

\[ B_3 = \{1,2,3\} \]

\[ \{1,2\} \quad \{1,3\} \quad \{2,3\} \]

\[ \{1\} \quad \{2\} \quad \{3\} \]

\[ \emptyset \]

\[ \mu(\emptyset) = \mu(\hat{\emptyset}) = 1, \]
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\} \]

\[ \{1, 2\} \{1, 3\} \{2, 3\} \]

\[ \{1\} \{2\} \{3\} \]

\[ \emptyset \]

\[ 1 \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\} \]

\[ B_3 = \{1, 2\}, \{1, 3\}, \{2, 3\} \]

\[ B_3 = \{1\}, \{2\}, \{3\} \]

\[ \emptyset, 1 \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]

\[ \mu(\{1\}) \]
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\} \]

\[ \{1, 2\} \]

\[ \{1, 3\} \]

\[ \{2, 3\} \]

\[ \{1\} \]

\[ \{2\} \]

\[ \{3\} \]

\[ \emptyset \]

\[ 1 \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]

\[ \mu(\{1\}) = -\mu(\emptyset) = -1, \]
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\} \]

\[
\begin{align*}
\{1, 2\} & \quad \{1, 3\} & \quad \{2, 3\} \\
\{1\} & \quad -1 & \quad \{2\} & \quad \{3\} \\
\emptyset & \quad 1
\end{align*}
\]

\[
\mu(\emptyset) = \mu(\hat{0}) = 1, \\
\mu(\{1\}) = -\mu(\emptyset) = -1,
\]
Example: The Boolean Algebra.

\[ B_3 = \begin{cases} \{1, 2, 3\} \\ \{1, 2\} \\ \{1\} \\ \emptyset \end{cases} \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]
\[ \mu(\{1\}) = -\mu(\emptyset) = -1, \]
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\} \]

\[ \{1, 2\} \]

\[ \{1, 3\} \]

\[ \{2, 3\} \]

\[ \{1\} \]

\[ -1 \]

\[ \{2\} \]

\[ -1 \]

\[ \{3\} \]

\[ -1 \]

\[ \emptyset \]

\[ 1 \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]

\[ \mu(\{1\}) = -\mu(\emptyset) = -1, \]

\[ \mu(\{1, 2\}) \]
Example: The Boolean Algebra.

\[
B_3 = \{1, 2, 3\} \\
\{1, 2\} \quad \{1, 3\} \quad \{2, 3\} \\
\{1\} \quad \{2\} \quad \{3\} \\
\emptyset \quad 1
\]

\[
\mu(\emptyset) = \mu(\hat{0}) = 1, \\
\mu(\{1\}) = -\mu(\emptyset) = -1, \\
\mu(\{1, 2\}) = -(\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\}))
\]
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\} \]

\[ \{1, 2\} \]

\[ \{1, 3\} \]

\[ \{2, 3\} \]

\[ \{1\} \]

\[ -1 \]

\[ \{2\} \]

\[ -1 \]

\[ \{3\} \]

\[ -1 \]

\[ \emptyset \]

\[ 1 \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]

\[ \mu(\{1\}) = -\mu(\emptyset) = -1, \]

\[ \mu(\{1, 2\}) = - (\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1, \]
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\}, \{1, 2\}, \{1\}, \{2\}, \{3\}, \emptyset \]

\[ \mu(\emptyset) = \mu(\hat{\emptyset}) = 1, \]
\[ \mu(\{1\}) = -\mu(\emptyset) = -1, \]
\[ \mu(\{1, 2\}) = -(\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1, \]
Example: The Boolean Algebra.

\[ B_3 = \begin{array}{cccc}
\{1, 2, 3\} & \{1, 2\} & \{1, 3\} & \{2, 3\} \\
\{1\} & -1 & 1 & -1 \\
\{2\} & 1 & -1 & 1 \\
\{3\} & -1 & 1 & -1 \\
\emptyset & 1 & 1 & 1 \\
\end{array} \]

\[
\mu(\emptyset) = \mu(\hat{0}) = 1, \\
\mu(\{1\}) = -\mu(\emptyset) = -1, \\
\mu(\{1, 2\}) = -(\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1,
\]
Example: The Boolean Algebra.

In $B_3$ we have

$\mu(\emptyset) = \mu(\hat{\emptyset}) = 1$,
$\mu(\{1\}) = -\mu(\emptyset) = -1$,
$\mu(\{1, 2\}) = -(\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1$,
$\mu(\{1, 2, 3\})$
Example: The Boolean Algebra.

\[ B_3 = \begin{array}{ccc}
\{1, 2, 3\} & 1 & \{1, 3\} & 1 & \{2, 3\} & 1 \\
\{1, 2\} & -1 & \{1, 3\} & 1 & \{2, 3\} & 1 \\
\{1\} & -1 & \{2\} & -1 & \{3\} & -1 \\
\emptyset & 1 & \end{array} \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]
\[ \mu(\{1\}) = -\mu(\emptyset) = -1, \]
\[ \mu(\{1, 2\}) = -(\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1, \]
\[ \mu(\{1, 2, 3\}) = -(1 - 1 - 1 - 1 + 1 + 1 + 1 + 1) = -1 \]
Example: The Boolean Algebra.

$$B_3 = \begin{array}{cccc}
\{1, 2, 3\} & -1 \\
\{1, 2\} & 1 & \{1, 3\} & 1 & \{2, 3\} & 1 \\
\{1\} & -1 & \{2\} & -1 & \{3\} & -1 \\
\emptyset & 1
\end{array}$$

$$\mu(\emptyset) = \mu(\hat{\emptyset}) = 1,$$
$$\mu(\{1\}) = -\mu(\emptyset) = -1,$$
$$\mu(\{1, 2\}) = - (\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1,$$
$$\mu(\{1, 2, 3\}) = -(1 - 1 - 1 - 1 + 1 + 1 + 1 + 1) = -1$$
Example: The Boolean Algebra.

\[
B_3 = \begin{array}{cccc}
\{1, 2, 3\} & -1 \\
\{1, 2\} & 1 & \{1, 3\} & 1 & \{2, 3\} & 1 \\
\{1\} & -1 & \{2\} & -1 & \{3\} & -1 \\
\emptyset & 1
\end{array}
\]

\[
\mu(\emptyset) = \mu(\emptyset) = 1, \\
\mu(\{1\}) = -\mu(\emptyset) = -1, \\
\mu(\{1, 2\}) = -(\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})) = -(1 - 1 - 1) = 1, \\
\mu(\{1, 2, 3\}) = -(1 - 1 - 1 - 1 + 1 + 1 + 1) = -1
\]

Conjecture

In \(B_n\) we have \(\mu(S) = (-1)^{|S|}\).
Example: The Divisor Lattice.

\[ D_{18} = \]

\[
\begin{array}{ccc}
  2 & 6 & 18 \\
  1 & 3 & 9 \\
  1 & 2 & 3 \\
\end{array}
\]

Conjecture

If \( d \in D_n \) has prime factorization \( d = p_{m_1} \cdots p_{m_k} \) then

\[
\mu(d) =
\begin{cases}
  (-1)^k & \text{if } m_1 = \ldots = m_k = 1, \\
  0 & \text{if } m_i \geq 2 \text{ for some } i.
\end{cases}
\]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
Example: The Divisor Lattice.

$$D_{18} =$$

$$\mu(1) = \mu(\hat{0}) = 1,$$

$$\mu(2) = \mu(3) = -1,$$
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
\[ \mu(6) \]
Example: The Divisor Lattice.

$$D_{18} =$$

\[
\begin{array}{cccc}
\bullet & 18 & \bullet & 9 \\
\bullet & 6 & \bullet & -1 \\
\bullet & 2 & \bullet & 3 \\
\bullet & 1 & \bullet & 1 \\
\end{array}
\]

$$\mu(1) = \mu(\hat{0}) = 1, \quad \mu(2) = \mu(3) = -1, \quad \mu(6) = -(\mu(1) + \mu(2) + \mu(3))$$
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ 18 \]

\[ 6 \]

\[ 9 \]

\[ 2 \]

\[ -1 \]

\[ 3 \]

\[ -1 \]

\[ 1 \]

\[ 1 \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]

\[ \mu(2) = \mu(3) = -1, \]

\[ \mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
\[ \mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
\[ \mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \]
\[ \mu(9) \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
\[ \mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \]
\[ \mu(9) = -(\mu(1) + \mu(3)) \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
\[ \mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \]
\[ \mu(9) = -(\mu(1) + \mu(3)) = -(1 - 1) = 0, \]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[
\begin{array}{c}
\mu(1) = \mu(\hat{0}) = 1, \\
\mu(2) = \mu(3) = -1, \\
\mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \\
\mu(9) = -(\mu(1) + \mu(3)) = -(1 - 1) = 0,
\end{array}
\]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
\[ \mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \]
\[ \mu(9) = -(\mu(1) + \mu(3)) = -(1 - 1) = 0, \]
\[ \mu(18) \]
Example: The Divisor Lattice.

\[
D_{18} = \begin{array}{ccc}
18 & 1 & 9 \\
6 & 1 & 3 \\
2 & -1 & 1
\end{array}
\]

\[
\mu(1) = \mu(\hat{0}) = 1,
\]
\[
\mu(2) = \mu(3) = -1,
\]
\[
\mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1,
\]
\[
\mu(9) = -(\mu(1) + \mu(3)) = -(1 - 1) = 0,
\]
\[
\mu(18) = -(1 - 1 - 1 + 1 + 0) = 0.
\]
Example: The Divisor Lattice.

\[ D_{18} = \]

\[
\begin{array}{ccc}
18 & 0 \\
6 & 1 & 9 \\
2 & -1 & 3 & -1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\mu(1) = \mu(\hat{0}) = 1,
\mu(2) = \mu(3) = -1,
\mu(6) = -(\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1,
\mu(9) = -(\mu(1) + \mu(3)) = -(1 - 1) = 0,
\mu(18) = -(1 - 1 - 1 + 1 + 0) = 0.
\]
**Example: The Divisor Lattice.**

\[ D_{18} = \]

\[ \begin{array}{c}
18 & 0 \\
2 & -1 \\
1 & 1 \\
6 & 1 \\
9 & 0
\end{array} \]

\[ \mu(1) = \mu(\hat{0}) = 1, \]
\[ \mu(2) = \mu(3) = -1, \]
\[ \mu(6) = - (\mu(1) + \mu(2) + \mu(3)) = -(1 - 1 - 1) = 1, \]
\[ \mu(9) = - (\mu(1) + \mu(3)) = -(1 - 1) = 0, \]
\[ \mu(18) = - (1 - 1 - 1 + 1 + 0) = 0. \]

**Conjecture**

*If* \( d \in D_n \) *has prime factorization* \( d = p_1^{m_1} \cdots p_k^{m_k} \)* then

\[ \mu(d) = \begin{cases} 
(-1)^k & \text{if } m_1 = \ldots = m_k = 1, \\
0 & \text{if } m_i \geq 2 \text{ for some } i.
\end{cases} \]
Theorem

1. If $f : P \to Q$ is an isomorphism and $x, y \in P$ then
   $$\mu_P(x, y) = \mu_Q(f(x), f(y)).$$
Theorem

1. If \( f : P \rightarrow Q \) is an isomorphism and \( x, y \in P \) then
   \[ \mu_P(x, y) = \mu_Q(f(x), f(y)). \]

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$$\mu(d) = \begin{cases} (-1)^k & \text{if } m_1 = \ldots = m_k = 1, \\ 0 & \text{if } m_i \geq 2 \text{ for some } i. \end{cases}$$
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The Incidence Algebra

The Möbius Function

The Möbius Inversion Theorem
Theorem (Möbius Inversion Theorem or MIT, Weisner (1935))

Consider a finite poset $P$ and two functions $f : P \rightarrow \mathbb{R}$ and $g : P \rightarrow \mathbb{R}$. Then the following are equivalent statements.

1. $f(y) = \sum_{x \leq y} g(x)$ for all $y \in P$. 

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Proof.

Let $L : x_1, \ldots, x_n$ be the linear extension used for $I(P)$.

Associate with $f$ the row vector $v_f = [f(x_1) \cdot \ldots \cdot f(x_n)]$ and similarly for $g$.

Then $f(y) = \sum_{x \leq y} g(x)$ for all $y \in P$ if and only if $v_f = v_g M \mu$ if and only if $g(y) = \sum_{x \leq y} \mu(x, y)f(x)$ for all $y \in P$. 

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Example: Theory of Finite Differences.
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Example: Theory of Finite Differences.

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\[ g(n) = \sum_{i \leq n} \mu(i, n)f(i) \]
Theorem (MIT)

\[
f(y) = \sum_{x \leq y} g(x) \quad \forall y \in P \iff g(y) = \sum_{x \leq y} \mu(x, y)f(x) \quad \forall y \in P. \quad \square
\]

Example: Theory of Finite Differences.

For \( g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R} \):

\[
\Delta g(n) = g(n) - g(n - 1), \quad S_g(n) = \sum_{i=0}^{n} g(i).
\]

Theorem (FTDC)

If \( g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R} \) then:

\[
\Delta S_g(n) = g(n).
\]

Proof. Consider the chain \( C_n \) and the restriction \( g : C_n \rightarrow \mathbb{R} \).

For each \( k \in C_n \), define

\[
f(k) = \sum_{i \leq k} g(i) = S_g(k).
\]

Then by the MIT applied to \( C_n \)

\[
g(n) = \sum_{i \leq n} \mu(i, n)f(i) = \mu(n, n)f(n) + \mu(n - 1, n)f(n - 1)
\]
Theorem (MIT)

\[ f(y) = \sum_{x \leq y} g(x) \forall y \in P \iff g(y) = \sum_{x \leq y} \mu(x, y)f(x) \forall y \in P. \]

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Theorem (Dual MIT)

\[ f(x) = \sum_{y \geq x} g(y) \forall x \in P \iff g(x) = \sum_{y \geq x} \mu(x, y)f(y) \forall x \in P. \ ]
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Example: Principle of Inclusion-Exclusion.
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Example: Principle of Inclusion-Exclusion.

Theorem (PIE)

Let \( U \) be a finite set and \( U_1, \ldots, U_n \subseteq U \).

\[ |U - \bigcup_{i=1}^{n} U_i| = |U| - \sum_{1 \leq i \leq n} |U_i| + \cdots + (-1)^n |\bigcap_{i=1}^{n} U_i|. \]
Theorem (Dual MIT)

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Proof. For the Boolean algebra \( B_n \), define \( f, g : B_n \rightarrow \mathbb{R} \) by

\[ f(S) = \text{# of elements in all } U_i, \ i \in S, \text{ and possibly other } U_j, \]
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Let \( U \) be a finite set and \( U_1, \ldots, U_n \subseteq U \).

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\]

Proof. For the Boolean algebra \( B_n \), define \( f, g : B_n \to \mathbb{R} \) by

\[
f(S) = \# \text{ of elements in all } U_i, i \in S, \text{ and possibly other } U_j,
g(S) = \# \text{ of elements in all } U_i, i \in S, \text{ and no other } U_j.
\]

Now \( f(S) = \left| \bigcap_{i \in S} U_i \right| \)
Theorem (Dual MIT)

\[ f(x) = \sum_{y \geq x} g(y) \forall x \in P \iff g(x) = \sum_{y \geq x} \mu(x, y)f(y) \forall x \in P. \]

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Let \( U \) be a finite set and \( U_1, \ldots, U_n \subseteq U \).

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\[ f(x) = \sum_{y \geq x} g(y) \forall x \in P \iff g(x) = \sum_{y \geq x} \mu(x, y) f(y) \forall x \in P. \]

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\]

Now \( f(S) = |\cap_{i \in S} U_i| \) and \( f(S) = \sum_{T \supseteq S} g(T) \). By the Dual MIT

\[
|U - \bigcup_{i=1}^{n} U_i| = g(\emptyset)
\]
Theorem (Dual MIT)

\[ f(x) = \sum_{y \geq x} g(y) \quad \forall x \in P \quad \iff \quad g(x) = \sum_{y \geq x} \mu(x, y)f(y) \quad \forall x \in P. \qed \]

Example: Principle of Inclusion-Exclusion.

Theorem (PIE)

Let \( U \) be a finite set and \( U_1, \ldots, U_n \subseteq U \).

\[ |U - \bigcup_{i=1}^{n} U_i| = |U| - \sum_{1 \leq i \leq n} |U_i| + \cdots + (-1)^n \sum_{i=1}^{n} |\bigcap_{i \in S} U_i|. \]

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\[
|U - \bigcup_{i=1}^{n} U_i| = g(\emptyset) = \sum_{T \supseteq \emptyset} \mu(\emptyset, T)f(T) = \sum_{T \in B_n} (-1)^{|T|} |\bigcap_{i \in T} U_i|. \]
Theorem (MIT)

\[ f(y) = \sum_{x \leq y} g(x) \quad \forall y \in P \iff g(y) = \sum_{x \leq y} \mu(x, y) f(x) \quad \forall y \in P. \]

Example: Number Theory

Theorem (Number Theory MIT)

Let \( f, g : \mathbb{Z}^+ \to \mathbb{R} \) satisfy \( f(n) = \sum_{d \mid n} g(d) \) for all \( n \in \mathbb{Z}^+ \).

Then \( g(n) = \sum_{d \mid n} \mu(n/d) f(d) \).

Proof. The restrictions \( f, g : D_n \to \mathbb{R} \) satisfy, for all \( m \in D_n \),

\[ f(m) = \sum_{d \mid m} g(d) = \sum_{d \leq D_n m} g(d) = \sum_{d \leq D_n n} \mu(n/d) f(d) = \sum_{d \mid n} \mu(n/d) f(d). \]

Apply the poset MIT to \( D_n \) and use \([d, n] \sim [1, n/d]\).
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Example: Number Theory

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Let \( f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R} \) satisfy \( f(n) = \sum_{d|n} g(d) \) for all \( n \in \mathbb{Z}_{>0} \).

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\[ g(n) = \sum_{d|n} \mu(n/d) f(d). \]
Theorem (MIT)

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Let \( f, g : \mathbb{Z}_0^+ \rightarrow \mathbb{R} \) satisfy \( f(n) = \sum_{d \mid n} g(d) \) for all \( n \in \mathbb{Z}_0^+ \). Then

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Apply the poset MIT to \( D_n \) and use \([d, n] \cong [1, n/d]\),
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