Möbius Functions of Posets I: Introduction to Partially Ordered Sets

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Motivating Examples

Poset Basics

Isomorphism and Products
Outline

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Isomorphism and Products
Example A: Combinatorics.

The Principle of Inclusion-Exclusion or PIE is a very useful tool in enumerative combinatorics.

Theorem (PIE)

Let $U$ be a finite set and $U_1, \ldots, U_n \subseteq U$.

$$|U - \bigcup_{i=1}^{n} U_i| = |U| - \sum_{1 \leq i \leq n} |U_i| + \sum_{1 \leq i < j \leq n} |U_i \cap U_j| - \cdots + (-1)^n |U_1 \cap \cdots \cap U_n|.$$
Example A: Combinatorics.
Given a set, \( S \), let

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Example B: Theory of Finite Differences.

Let $Z \geq 0$ be the nonnegative integers. If one takes a function $f: Z \geq 0 \to \mathbb{R}$ then there is an analogue of the derivative, namely the difference operator

$$\Delta f(n) = f(n) - f(n-1)$$

(where $f(-1) = 0$ by definition).

There is also an analogue of the integral, namely the summation operator

$$Sf(n) = \sum_{i=0}^{n} f(i).$$

The Fundamental Theorem of the Difference Calculus or FTDC is as follows.

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If $f: Z \geq 0 \to \mathbb{R}$ then

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Theorem (FTDC)
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If \( d, n \in \mathbb{Z} \) then write \( d \mid n \) if \( d \) divides evenly into \( n \).

The number-theoretic M"obius function is \( \mu : \mathbb{Z}_{>0} \to \mathbb{Z} \) defined as

\[
\mu(n) = \begin{cases} 
0 & \text{if } n \text{ is not square free,} \\
\left( -1 \right)^k & \text{if } n \text{ is the product of } k \text{ distinct primes.}
\end{cases}
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The importance of \( \mu \) lies in the number-theoretic M"obius Inversion Theorem or MIT.

**Theorem (Number Theory MIT)**

Let \( f, g : \mathbb{Z}_{>0} \to \mathbb{R} \) satisfy

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f(n) = \sum_{d \mid n} g(d)
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for all \( n \in \mathbb{Z}_{>0} \). Then

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g(n) = \sum_{d \mid n} \mu(n/d) f(d).
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If $d, n \in \mathbb{Z}$ then write $d|n$ if $d$ divides evenly into $n$. 

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Möbius inversion over partially ordered sets is important for the following reasons.

1. It unifies and generalizes the three previous examples.
2. It makes the number-theoretic definition transparent.
3. It encodes topological information about partially ordered sets.
4. It can be used to solve combinatorial problems.
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Poset Basics

Isomorphism and Products
A *partially ordered set* or *poset* is a set $P$ together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Given any poset notation, if we wish to be specific about the poset $P$ involved, we attach $P$ as a subscript. For example, using $\leq_P$ for $\leq$.

We also adopt the usual conventions for inequalities. For example, $x < y$ means $x \leq y$ and $x \neq y$.

If $x, y \in P$ then $x$ is covered by $y$ or $y$ covers $x$, written $x \overset{\downarrow}{\leq} y$, if $x < y$ and there is no $z$ with $x < z < y$.

The *Hasse diagram* of $P$ is the (directed) graph with vertices $P$ and an edge from $x$ up to $y$ if $x \overset{\downarrow}{\leq} y$. 
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If $x, y \in P$ then $x$ is covered by $y$ or $y$ covers $x$, written $x \prec y$, if $x < y$ and there is no $z$ with $x < z < y$. 

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If $x, y \in P$ then $x$ is covered by $y$ or $y$ covers $x$, written $x \triangleleft y$, if $x < y$ and there is no $z$ with $x < z < y$. 
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The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ with the usual $\leq$ on the integers.
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Example: The Boolean Algebra.

The Boolean algebra $B_n = \{ S : S \subseteq \{1, 2, \ldots, n\} \}$ is partially ordered by $S \leq T$ if and only if $S \subseteq T$.

$B_3 = \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

Note that $B_3$ looks like a cube.
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Example: The Divisor Lattice.

Given $n \in \mathbb{Z} > 0$ the corresponding divisor lattice $D_n = \{d \in \mathbb{Z} > 0 : d \mid n\}$ is partially ordered by $c \leq D_n d$ if and only if $c \mid d$.

$D_{18}$ looks like a rectangle.
Example: The Divisor Lattice.
Given $n \in \mathbb{Z}_{>0}$ the corresponding \textit{divisor lattice} is

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![Diagram](image)
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2 3 6 9
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D_{18} = \begin{array}{c}
\bullet & 18 \\
\bullet & 12 \\
\bullet & 6 \\
\bullet & 2 \\
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In a poset $P$, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. 

Example. The poset on the left has minimal elements $u$ and $v$, and maximal elements $x$ and $y$. 

A poset has a zero if it has a unique minimal element, $\hat{0}$. A poset has a one if it has a unique maximal element, $\hat{1}$. A poset if bounded if it has both a $\hat{0}$ and a $\hat{1}$. 

Example. Our three fundamental examples are bounded: $\hat{0} C_n = 0$, $\hat{1} C_n = n$, $\hat{0} B_n = \emptyset$, $\hat{1} B_n = \{1, \ldots, n\}$, $\hat{0} D_n = 1$, $\hat{1} D_n = n$. 

If $x \leq y$ in $P$ then the corresponding closed interval is $[x, y] = \{z : x \leq z \leq y\}$. 

Open and half-open intervals are defined analogously. 

Note that $[x, y]$ is a poset in its own right and it has a zero and a one: $\hat{0} [x, y] = x$, $\hat{1} [x, y] = y$. 
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\textbf{Example.} The poset on the left has minimal elements $u$ and $v$, and maximal elements $x$ and $y$.

A poset \textit{has a zero} if it has a unique minimal element, $\hat{0}$. A poset \textit{has a one} if it has a unique maximal element, $\hat{1}$. A poset \textit{is bounded} if it has both a $\hat{0}$ and a $\hat{1}$.
In a poset $P$, a \textit{minimal} element is $x \in P$ such that there is no $y \in P$ with $y < x$. A \textit{maximal} element is $x \in P$ such that there is no $y \in P$ with $y > x$.

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$\hat{0}_{C_n} = 0$, $\hat{1}_{C_n} = n$, $\hat{0}_{B_n} = \emptyset$. 
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**Example.** Our three fundamental examples are bounded:

$\hat{0}_{C_n} = 0$, $\hat{1}_{C_n} = n$, $\hat{0}_{B_n} = \emptyset$, $\hat{1}_{B_n} = \{1, \ldots, n\}$, $\hat{0}_{D_n} = 1$, $\hat{1}_{D_n} = n$. 

![Diagram showing a poset with minimal and maximal elements](image)
In a poset $P$, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

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If $x \leq y$ in $P$ then the corresponding **closed interval** is

$$[x, y] = \{z : x \leq z \leq y\}.$$
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$\hat{0}_{Cn} = 0$, $\hat{1}_{Cn} = n$, $\hat{0}_{Bn} = \emptyset$, $\hat{1}_{Bn} = \{1, \ldots, n\}$, $\hat{0}_{Dn} = 1$, $\hat{1}_{Dn} = n$.

If $x \leq y$ in $P$ then the corresponding *closed interval* is 

$$[x, y] = \{z : x \leq z \leq y\}.$$ 

Open and half-open intervals are defined analogously.
In a poset \( P \), a **minimal** element is \( x \in P \) such that there is no \( y \in P \) with \( y < x \). A **maximal** element is \( x \in P \) such that there is no \( y \in P \) with \( y > x \).

**Example.** The poset on the left has minimal elements \( u \) and \( v \), and maximal elements \( x \) and \( y \).

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**Example.** Our three fundamental examples are bounded:
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\hat{0}_{C_n} = 0, \quad \hat{1}_{C_n} = n, \quad \hat{0}_{B_n} = \emptyset, \quad \hat{1}_{B_n} = \{1, \ldots, n\}, \quad \hat{0}_{D_n} = 1, \quad \hat{1}_{D_n} = n.
\]

If \( x \leq y \) in \( P \) then the corresponding **closed interval** is
\[
[x, y] = \{ z : x \leq z \leq y \}.
\]

Open and half-open intervals are defined analogously. Note that \([x, y]\) is a poset in its own right and it has a zero and a one:
\[
\hat{0}_{[x,y]} = x, \quad \hat{1}_{[x,y]} = y.
\]
Example: The Chain.
In $C_9$ we have the interval $[4, 7]$:
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![Diagram showing the interval [4, 7] in $C_9$.]

This interval looks like $C_3$. 
Example: The Boolean Algebra.
In $B_7$ we have the interval $[\{3\}, \{2, 3, 5, 6\}]$: 
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```
{2, 3, 5}   {2, 3, 6}   {3, 5, 6}
{2, 3}     {3, 5}     {3, 6}
{3}
```
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```
{2, 3, 5, 6}

{2, 3, 5}  {2, 3, 6}  {3, 5, 6}

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{3}
```
Example: The Boolean Algebra.

In $B_7$ we have the interval $[\{3\}, \{2, 3, 5, 6\}]$:

\[
\begin{align*}
\{2, 3, 5, 6\} \\
\{2, 3, 5\} & \quad \{2, 3, 6\} & \quad \{3, 5, 6\} \\
\{2, 3\} & \quad \{3, 5\} & \quad \{3, 6\} \\
\{3\} \\
\end{align*}
\]

Note that this interval looks like $B_3$. 
Example: The Divisor Lattice.

In $D_{80}$ we have the interval $[2, 40]$: 
Example: The Divisor Lattice.
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\[
\begin{array}{c}
2 \\
\end{array}
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In $D_{80}$ we have the interval $[2, 40]$:

Note that this interval looks like $D_{18}$. 
If $P$ is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \land y$ in $P$ such that

1. $x \land y \leq x$ and $x \land y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \land y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \lor y$ in $P$ such that

1. $x \lor y \geq x$ and $x \lor y \geq y$,
2. if $z \geq x$ and $z \geq y$ then $z \geq x \land y$.

We say $P$ is a *lattice* if every $x, y \in P$ have both a meet and a join.

Example.

1. $C_n$ is a lattice with $i \land j = \min\{i, j\}$ and $i \lor j = \max\{i, j\}$.

2. $B_n$ is a lattice with $S \land T = S \cap T$ and $S \lor T = S \cup T$.

3. $D_n$ is a lattice with $c \land d = \gcd\{c, d\}$ and $c \lor d = \lcm\{c, d\}$.
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If $P$ is a poset then $x, y \in P$ have a \textit{greatest lower bound} or \textit{meet} if there is an element $x \wedge y$ in $P$ such that

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If $P$ is a poset then $x, y \in P$ have a greatest lower bound or **meet** if there is an element $x \land y$ in $P$ such that

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Outline

Motivating Examples

Poset Basics

Isomorphism and Products
For posets $P$ and $Q$, an \textit{order preserving map} is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An \textit{isomorphism} is a bijection $f : P \to Q$ such that both $f$ and $f^{-1}$ are order preserving. In this case $P$ and $Q$ are isomorphic, written $P \sim Q$.

**Proposition**

If $i \leq j$ in $C^n$ then $[i, j] \sim C^{j-i}$.

If $S \subseteq T$ in $B^n$ then $[S, T] \sim B_{|T|-S}$.

If $c \mid d$ in $D^n$ then $[c, d] \sim D_{d/c}$.

**Proof for $C^n$**.

Define $f : [i, j] \to C^{j-i}$ by $f(k) = k - i$.

Then $f$ is order preserving since $k \leq l \implies k - i \leq l - i \implies f(k) \leq f(l)$.

Also $f$ is bijective with inverse $f^{-1}(k) = k + i$.

It is easy to check that $f^{-1}$ is order preserving.

**Exercise.**

Prove the other two parts of the Proposition.
For posets $P$ and $Q$, an order preserving map is $f : P \to Q$ with

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An isomorphism is a bijection $f : P \to Q$ such that both $f$ and $f^{-1}$ are order preserving. In this case $P$ and $Q$ are isomorphic, written $P \cong Q$. 

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If $i \leq j$ in $C_n$ then $[i, j] \cong C_{j-i}$. 

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For posets $P$ and $Q$, an order preserving map is $f : P \rightarrow Q$ with
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An isomorphism is a bijection $f : P \rightarrow Q$ such that both $f$ and $f^{-1}$ are order preserving. In this case $P$ and $Q$ are isomorphic, written $P \cong Q$.

**Proposition**

If $i \leq j$ in $C_n$ then $[i, j] \cong C_{j-i}$.
If $S \subseteq T$ in $B_n$ then $[S, T] \cong B_{|T-S|}$.
If $c|d$ in $D_n$ then $[c, d] \cong D_{d/c}$.

**Proof for $C_n$.** Define $f : [i, j] \rightarrow C_{j-i}$ by $f(k) = k - i$. Then $f$ is order preserving since
\[ k \leq l \implies k - i \leq l - i \implies f(k) \leq f(l). \]

Also $f$ is bijective with inverse $f^{-1}(k) = k + i$. It is easy to check that $f^{-1}$ is order preserving. 

**Exercise.** Prove the other two parts of the Proposition.
If $P$ and $Q$ are posets, then their *product* is

$$P \times Q = \{(a, x) : a \in P, \ x \in Q\}$$

partially ordered by

$$(a, x) \leq_{P \times Q} (b, y) \iff a \leq_P b \text{ and } x \leq_Q y.$$
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If \( P \) and \( Q \) are posets, then their \textit{product} is

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*For the Boolean algebra: $B_n \cong (C_1)^n$.***
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*For the Boolean algebra: $B_n \cong (C_1)^n$. If the prime factorization of $n$ is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.***
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Exercise. Prove the statement for $D_n$. 
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