Möbius Functions of Lattices

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THE NBB THEOREM

Let $(L, \leq)$ be a finite lattice with minimum $\hat{0}$ and maximum $\hat{1}$. Let $\mu : L \to \mathbf{Z}$ be L's Möbius function which is the unique function satisfying

$$\sum_{y \leq x} \mu(y) = \delta_{\hat{0}x}.$$

Let $A(L)$ be the atom set of L and put an arbitrary partial order $\trianglelefteq$ on $A(L)$. Then $D \subseteq A(L)$ is *bounded below (BB)* if, for every $d \in D$ there is an $a \in A(L)$ such that

$$\begin{aligned} a &\triangleleft d \qquad \text{and} \\ a &< \bigvee D. \end{aligned}$$

Then $B \subseteq A(L)$ is an *NBB base of* x if $x = \bigvee B$ and B does not contain any D which is BB.

Theorem 1 *Let* L *be any finite lattice and let* $\trianglelefteq$ *be any partial order on* $A(L)$*. Then for all* $x \in L$

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all NBB bases B *of* x*.*

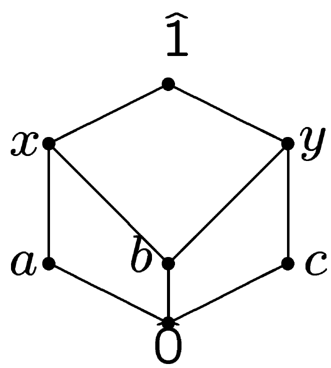
$D \subseteq A(L)$ is *BB* if $\forall d \in D$ $\exists a \in A(L)$ such that

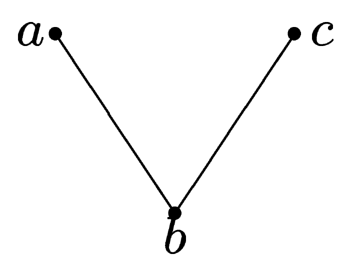
$$a \lhd d \qquad \text{and} \qquad a < \bigvee D.$$

NBB Theorem. For all $x \in L$

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all NBB bases B of x.

$(L, \leq) =$ 

$(A(L), \unlhd) =$ 

Ex. Note that from the definition of BB

1. No set containing a min. element of $\unlhd$ is BB.
2. No single element set is BB.

So for L and $\unlhd$ in the figure, the only possible BB set is $\{a, c\}$. It is since $b \lhd a, c$ and $b < \bigvee\{a, c\} = \hat{1}$.

x	$\hat{0}$	a	x	$\hat{1}$
NBB bases	$\emptyset$	a	ab	none
$\mu(x)$	$(-1)^0$	$(-1)^1$	$(-1)^2$	0

NBB Theorem. For all $x \in L$

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all NBB bases B of x.

Pf. Let $\tilde{\mu}(x) = \sum_B (-1)^{|B|}$ & show $\sum_{y \le x} \tilde{\mu}(y) = \delta_{\hat{0}x}$.

$x = \hat{0}$: NBB base $B = \emptyset$ & $\sum_{y \le \hat{0}} \tilde{\mu}(y) = (-1)^0 = 1$.

$x > \hat{0}$: want signed $\mathcal{S}$ & sign-reversing involution ι.

$$\begin{aligned} \mathcal{S} &:= \{B \text{ an NBB base for some } y \le x\} \\ \epsilon(B) &:= (-1)^{|B|} \\ \sum_{y \le x} \tilde{\mu}(y) &= \sum_{B \in \mathcal{S}} \epsilon(B). \\ \iota(B) &:= B \triangle a_0 \end{aligned}$$

where $\triangle$ is the symmetric difference operator and $a_0 \le x$ is $\trianglelefteq$-min. Suffices to show $\iota(B)$ is still NBB.

$\iota(B) = B \setminus a_0$: clear. If $\iota(B) = B \cup a_0 := B' \supseteq D$ where D is BB then $a_0 \in D$. Let a be the element guaranteed for a_0 from the definition of BB. Then $a \triangleleft a_0$ and $a < \bigvee D \le \bigvee B' \le x$, contradicting the definition of a_0. ∎

COMPARISON WITH NBC AND CROSSCUT

Let L be geometric with rank function ρ. So for any $B \subseteq A(L)$ we have $\rho(\bigvee B) \leq |B|$. Say B is *independent* if $\rho(\bigvee B) = |B|$. A minimal dependent set C is a *circuit*. If $\trianglelefteq$ is a total order on $A(L)$ then $C' = C \setminus \min C$ is a *broken circuit (BC)*. An *NBC base B for x* has $\vee B = x$ and B contains no BC.

Theorem 2 (Rota) *1. (NBC) Let L be geometric, $\trianglelefteq$ total. If $x \in L$:*

$$\mu(x) = (-1)^{\rho(x)}(\textit{number of NBC bases of } x).$$

2. (Crosscut) Let L be any finite lattice. If $x \in L$:

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all $B \subseteq A(L)$ with $\bigvee B = x$.

If L is geometric and $\trianglelefteq$ is total than the NBC and NBB bases are the same. Further, all bases of x have size $\rho(x)$. If L is arbitrary and $\trianglelefteq$ is total incomparability then the NBB bases are all B with $\bigvee B = x$. Thus our theorem interpolates between NBC and Crosscut.

APPLICATION 1: SHUFFLE POSETS

A *subword* of $\mathbf{x} = x_1 \ldots x_m$ is word $\mathbf{x}' = x_{i_1} \ldots x_{i_k}$ with $i_1 < \ldots < i_k$. A *shuffle* of $\mathbf{x}$ and $\mathbf{y} = y_1 \ldots y_n$ where $\mathbf{x} \cap \mathbf{y} = \emptyset$ is $\mathbf{s} = s_1 \ldots s_{m+n}$ having $\mathbf{x}$ and $\mathbf{y}$ as subwords denoted $\mathbf{s}_\mathbf{x}$ and $\mathbf{s}_\mathbf{y}$

Fix $\mathbf{x} = x_1 \ldots x_m$, $\mathbf{y} = y_1 \ldots y_n$. Greene's *poset of shuffles* has as elements all shuffles $\mathbf{w}$ of a subword of $\mathbf{x}$ and a subword of $\mathbf{y}$ with $\mathbf{v} \leq \mathbf{w}$ iff $\mathbf{v}_\mathbf{x} \supseteq \mathbf{w}_\mathbf{x}$ and $\mathbf{v}_\mathbf{y} \subseteq \mathbf{w}_\mathbf{y}$.

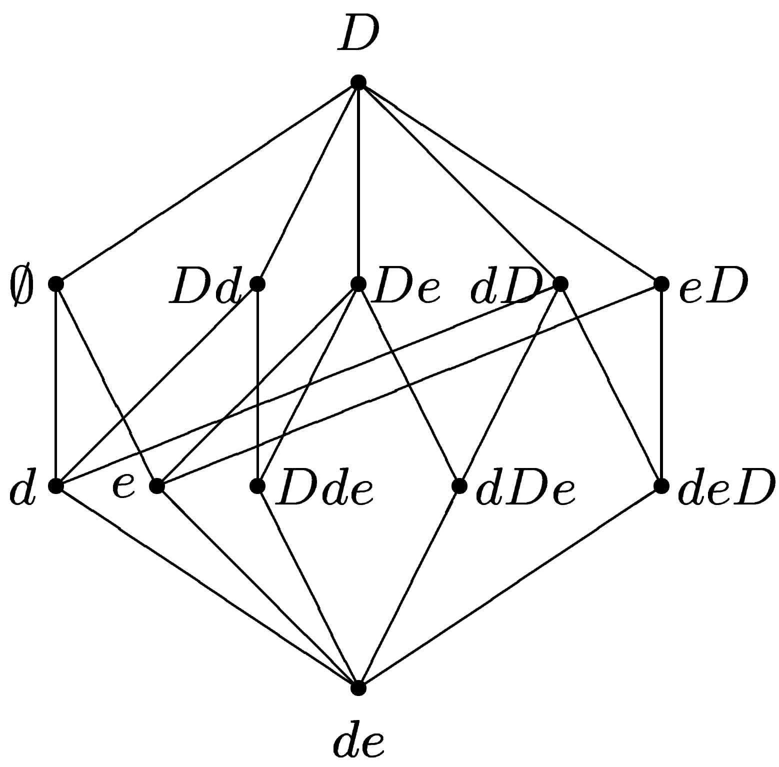
$A(\mathcal{W}_{m,n})$: An *a-atom*, resp. *b-atom*, is gotten from $\mathbf{x}$ by deleting a letter of $\mathbf{x}$, resp. inserting a letter of $\mathbf{y}$. Let $A_a =$ set of a-atoms, $A_b =$ set of b-atoms. Define $\trianglelefteq$ by $\mathbf{a} \triangleleft \mathbf{b}$ iff $\mathbf{a} \in A_a$ and $\mathbf{b} \in A_b$

Theorem 3 *1. Let* $\mathbf{s}$ *be a shuffle of* $\mathbf{x}, \mathbf{y}$ *and let*

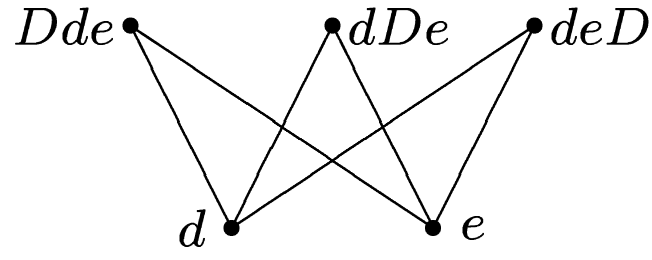
$$B_\mathbf{s} = A_a \cup \{\mathbf{b} \in A_b \ : \ \mathbf{b} \leq \mathbf{s}\}.$$

Then the NBB bases of $\mathbf{y} \in \mathcal{W}_{m,n}$ *under the given partial order are exactly the* $B_\mathbf{s}$.

2. [Greene] $\mu(\mathcal{W}_{m,n}) = (-1)^{m+n}\binom{m+n}{m}$. ■



$\mathcal{W}_{2,1}$



$(A(\mathcal{W}_{2,1}), \trianglelefteq)$

Example $\mathcal{W}_{2,1}$**:** $A_a = \{d,\ e\}$, $A_b = \{Dde,\ dDe,\ deD\}$

s	Dde	dDe	deD
$B_{\rm s}$	$\{d, e, Dde\}$	$\{d, e, dDe\}$	$\{d, e, deD\}$

$D = \{Dde, dDe\}$ is BB: $\bigvee D = De$ so $e < \bigvee D$ and $e \triangleleft Dde, dDe$.

$B = \{d, e, Dde\}$ is NBB: Take $D \subseteq B$. No D with $d \in D$ is BB (d is $\trianglelefteq$-min) nor with $|D| \leq 1$. So check $D = \{e, Dde\}$: $\bigvee D = De$ and e in $\trianglelefteq$-min under De so D is not BB.

2: NON-CROSSING PARTITIONS

Arrange the numbers in $[n] = \{1, \ldots, n\}$ in order around a circle. Partition $\pi = B_1/\ldots/B_k$ of $[n]$ is *non-crossing* if, when replacing each block by a complete graph, no edge of B_i crosses an edge of B_j. Let NC_n be the lattice of non-crossing partitions ordered by refinement.

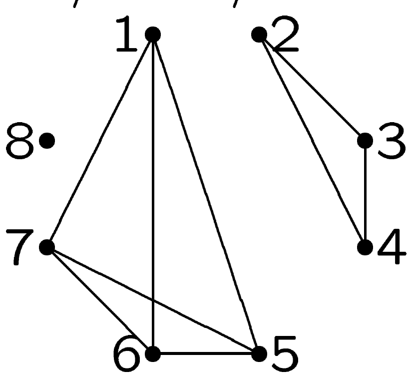
Each atom of NC_n is an edge rs. A set of atoms is *non-crossing* if its graph contains no pair of crossing edges. Define $\trianglelefteq$ to be the ranked poset with rank $r-1$ being $\{rs : r > s\}$ and all possible covers between ranks.

Theorem 4 *1. The NBB bases of NC_n are those non-crossing forests obtained by picking at most one atom from each rank of $(A(NC_n), \trianglelefteq)$.*

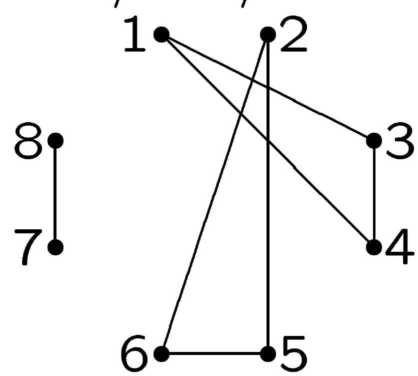
2. [Kreweras] $\mu(\hat{1}) = (-1)^{n-1} C_{n-1}$ where C_{n-1} is a Catalan number. ■

Examples.

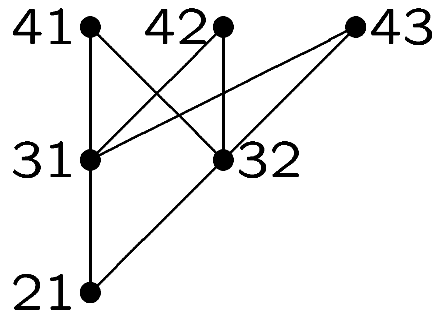
noncrossing partition 8/7651/432:



crossing partition 87/652/431:



The poset $(A(NC_4), \trianglelefteq)$:



Some BB and NBB sets in $A(NC_4)$:

1. $D = \{31, 32\}$ is BB (atoms from the same rank) since we have $21 \triangleleft 31, 32$ and $21 < \bigvee D = 321$.

2. $D = \{31, 42\}$ is BB (crossing atoms) since we have $21 \triangleleft 31, 42$ and $21 < \bigvee D = 4321$.

3. $B = \{32, 42\}$ is NBB since if $b = 32$ then $a \triangleleft b$ implies $a = 21$. But 21 is not $\triangleleft \bigvee B = 432$.

LL LATICES & SUPERSOLVABILITY

Let $\Delta : \hat{0} = x_0 < x_1 < \ldots < x_{n-1} < x_n = \hat{1}$ be a maximal chain of L. It induces both *levels*

$$A_i = \{a \in A \mid a \leq x_i \text{ but } a \not\leq x_{i-1}\}$$

and $\trianglelefteq$ by $a \triangleleft b$ iff $a \in A_i$ and $b \in A_j$ with $i < j$.

$$\rho(x) := \#\{i \ : \ A_i \text{ contains an atom } \leq x\}.$$

The *level condition* is

$\trianglelefteq$ is induced & $a \triangleleft b_1 \triangleleft b_2 \triangleleft \ldots \triangleleft b_k \Rightarrow a \not\leq \bigvee_{i=1}^{k} b_i$.

An element $x \in L$ is *left-modular* if for all $z \leq y$:

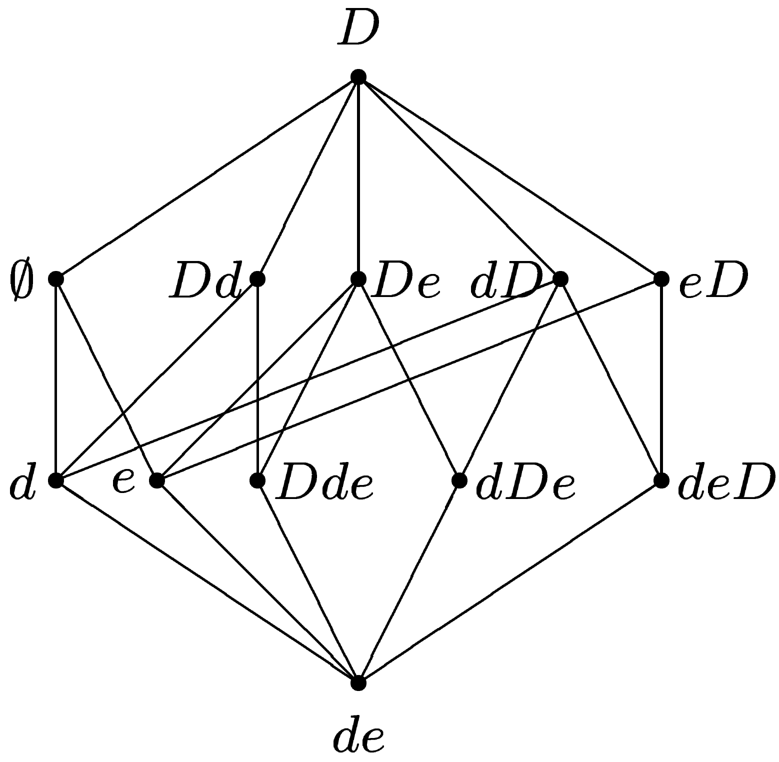
$$z \vee (x \wedge y) = (z \vee x) \wedge y$$

Chain Δ/lattice L are *left-modular* if all the elements x_i are left-modular. An *LL lattice* has a left-modular Δ satisfying the level condition.

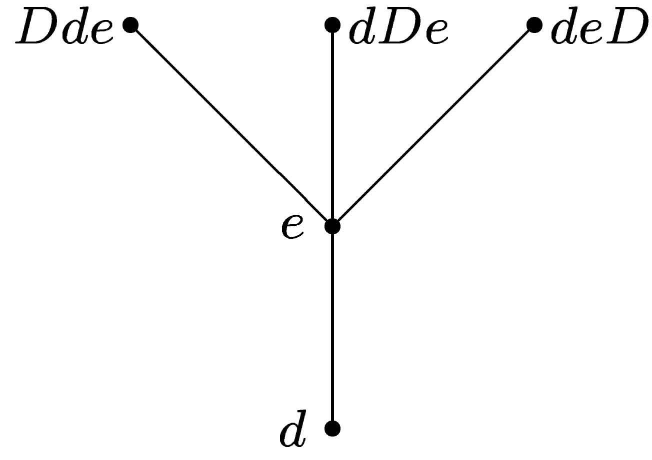
Proposition 5 *The following implications hold for L but not their converses.*

1. *semimodular $\Rightarrow$ level condition*

2. *supersolvable $\Rightarrow$ left-modular.* ∎



$\mathcal{W}_{2,1}$



$(A(\mathcal{W}_{2,1}), \trianglelefteq)$

Example $\mathcal{W}_{2,1}$: Let $\Delta : de < d < \emptyset < D$ of length 3. This induces

$$A_1 = \{d\},\ A_2 = \{e\},\ A_3 = \{Dde, dDe, deD\}$$

and the partial order in the figure. Bases for $\hat{1}$:

$$\{d, e, Dde\},\ \{d, e, dDe\},\ \{d, e, deD\}$$

Characteristic polynomial

$$\begin{aligned} \chi(\mathcal{W}_{2,1}, t) &:= \sum_{x \in \mathcal{W}_{2,1}} \mu(x) t^{3-\rho(x)} \\ &= (t-1)^2(t-3) \\ &= (t - |A_1|)(t - |A_2|)(t - |A_3|) \end{aligned}$$

Theorem 6 *Let* (L, Δ) *be LL with* Δ *of length* n *and* $\trianglelefteq$ *induced. Then*

1. The NBB bases of L *are those taking* ≤ 1 *atom from each* A_i *and* B *an NBB base of* $x \Rightarrow |B| = \rho(x)$.

2. $\chi(L, t) := \sum_{x \in L} \mu(x) t^{n-\rho(x)} = \prod_{i=1}^{n}(t - |A_i|)$. ■

Proof. We will show 1 ⇒ 2. By the NBB Theorem:

$$\begin{aligned} \chi(L,t) &= \sum_{x \in L} \sum_{\bigvee\{B:\text{NBB}\}=x} (-1)^{|B|} t^{n-\rho(x)} \\ &= \sum_{|B \cap A_i| \leq 1} (-1)^{|B|} t^{n-|B|} \\ &= \prod_{i=1}^{n} (t - |A_i|). \quad \blacksquare \end{aligned}$$

Corollary 7 *Consider the shuffle posets* $\mathcal{W}_{m,n}$ *with* $\mathbf{x} = x_1 \ldots x_m$, $\mathbf{y} = y_1 \ldots y_n$.

1. [Greene] There is a left-modular chain

$$\Delta : \mathbf{x} < x_2 \ldots x_n < \ldots < \emptyset < y_1 < y_1 y_2 < \ldots < \mathbf{y}$$

2. If $n = 1$ *then* Δ *satifies the level condition so*

$$\chi(\mathcal{W}_{m,1}, t) = (t-1)^m (t - m - 1).$$

For general $n > 2$, Δ *does not satisfy the level condition and* χ *does not factor.* ■

REMARKS

Topology and algebra. Segev has shown that the NBB complex is homotopic to the order complex. This can be used to rederive results of Kahn, Linusson, Edelman and Reiner. Liu and S have shown that left-modular lattices are shellable. Can NBB sets be used to define an Orlik-Solomon algebra?

Perfect orders. When computing $\mu(x)$ it is simplest to have the minimum number of bases, namely $|\mu(x)|$, all of the same parity. Call $\trianglelefteq$ *perfect* if this happens for alll $x \in L$. If $\trianglelefteq$ is perfect then so is any linear extension, however it is often clearer combinatorially to use $\trianglelefteq$ with as few relations as possible. There are also posets with no perfect order. Can one characterize which posets have a perfect order?

Spliting χ. Stanley has proved that if $x \in L$ with x modular and L geometric then

$$\chi(L,t) = \chi(L_x,t) \sum_{y : y \wedge x = \hat{0}} \mu(y) t^{n-\rho(x)-\rho(y)}$$

where L_x is the lower order ideal generated by x. An analog of this result is currently being investigated by Liu and S.