The Incidence Algebra of a Composition Poset

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Compositions

Rational generating functions

Commuting variables

The zeta and Möbius functions

Comments and open problems
Outline

Compositions

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Comments and open problems
Let $\mathbb{P}$ be the positive integers.

A composition of a non-negative integer $N$ is a sequence $w = k_1 k_2 \ldots k_r$ with all $k_i \in \mathbb{P}$ and $\sum i k_i = N$.

Let $c_N$ be the number of compositions of $N$.

Ex. If $N = 3$ then $c_3 = 4$ counting compositions $3, 21, 12, 111$.

Theorem $c_N = \begin{cases} 2^{N-1} & \text{if } N \geq 1 \\ 1 & \text{if } N = 0 \end{cases}$.

So we have the rational generating function $\sum_{N \geq 0} c_N x^N = \frac{1}{1 - 2x}$.

Question: Is this an isolated incident or part of a larger picture?
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A *composition* of a non-negative integer \( N \) is a sequence

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Any set $A$ (the alphabet) has Kleene closure

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Partially order $P^*$ (Bergeron, Bousquet-Mélou, and Dulucq, 1995): If $u = k_1 \ldots k_r$ and $w = l_1 \ldots l_s$ then $u \leq w$ iff there is a subsequence $l_{i_1} \ldots l_{i_r}$ of $w$ with

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\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline \\
4 & 1 & 4 & 3 & 2 & 4 & 2
\end{array}
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$I = \{4, 5, 6\}$.  

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$$w = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 1 & 4 & 3 & 2 & 4 & 2 \end{array} \quad \text{and} \quad I = \{3, 5, 6\}. 

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\text{u} & = & 4 & 1 & 3 & & & \\
\end{array}
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and $I = \{3, 5, 6\}$. Given $u \leq w$ there is a unique rightmost embedding, $I$, such that $I \geq I'$ componentwise for all embeddings $I'$. The embedding above is rightmost.
\( \mathbb{P}^* = \epsilon \)
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\[
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Comments and open problems
For any alphabet $A$, the \textit{formal power series in noncommuting variables} $A$ \textit{with integral coefficients} is

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\mathbb{Z}\langle\langle A \rangle\rangle = \left\{ f = \sum_{w \in A^*} c(w)w \mid c(w) \in \mathbb{Z} \quad \forall w \right\}.
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Let $[n] = \{1, \ldots, n\}$ have alphabet $[\bar{n}] = \{\bar{1}, \ldots, \bar{n}\}$. Given $u \in [\bar{n}]^*$, consider $Z(u) = \sum_{w \geq u} w \in \mathbb{Z}\langle\langle [\bar{n}] \rangle\rangle$.

\textbf{Ex.} $Z(\bar{1}\bar{1}) = \bar{1}\bar{1} + \bar{1}\bar{1}\bar{1} + \bar{1}\bar{2} + \bar{2}\bar{1} + \cdots$

\textbf{Theorem (Björner & S)} For all $u \in [\bar{n}]^*$, the series $Z(u)$ is rational.

Given $f = \sum_{w \in A^*} c(w)w$ with $c(\epsilon) = 0$, let $f^* = \epsilon + f + f^2 + f^3 + \cdots = (\epsilon - f)^{-1}$.

\textbf{Convention:} If $S \subseteq A$, then we also let $S$ stand for $\sum_{s \in S} s$. 
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Let $[n] = \{1, \ldots, n\}$ have alphabet $[\tilde{n}] = \{\tilde{1}, \ldots, \tilde{n}\}$. Given $u \in [\tilde{n}]^*$, consider

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Proof  We generate each $w \geq u$ by rightmost embedding as follows.

Ex. If $n = 4$ and $k = 3$ then $z(\bar{3}) = (\bar{3} + \bar{4})(\bar{1} + \bar{2}) \cdot \cdots$
Theorem (Björner & S)

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z(k) = [k, \bar{n}][k-1]^*
\]

where \([k, n] = \{k, k + 1, \ldots, n\}\).
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Now if \( u = \bar{k}_1 \ldots \bar{k}_r \) then

\[
Z(u) = [\bar{n}]^* z(\bar{k}_1) \cdots z(\bar{k}_r).
\]

**Ex.** If \( n = 4 \) and \( k = 3 \) then

\[
z(\bar{3}) = (\bar{3} + 4)(\bar{1} + \bar{2})^* = \bar{3} + 4 + 3 \bar{1} + 3 \bar{2} + 4 \bar{1} + 4 \bar{2} + \cdots
\]
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Recall:

\[ Z(u) = [\bar{n}]^* z(\bar{k}_1) \cdots z(\bar{k}_r) \quad \text{with} \quad z(\bar{k}) = [\bar{k}, \bar{n}][\bar{k} - 1]^*. \]
Recall:
\[ Z(u) = [\bar{n}]^*z(\bar{k}_1) \cdots z(\bar{k}_r) \quad \text{with} \quad z(\bar{k}) = [\bar{k}, \bar{n}][\overline{k - 1}]^*. \]

The *norm* of \( u = \bar{k}_1 \ldots \bar{k}_r \in \mathbb{P}^* \) is \( |u| = \sum_i k_i \).
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Let $x$ be a variable and substitute $\bar{k} \sim x^k$. 
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\[
\begin{align*}
  u = \bar{k}_1 \ldots \bar{k}_r & \sim x^{k_1} \ldots x^{k_r} = x^{|u|}, \\
  z(\bar{k}) & \sim (x^k + x^{k+1} + \cdots + x^n)(x + x^2 + \cdots + x^{k-1})^* \\
  & = \frac{x^k + x^{k+1} + \cdots + x^n}{1 - (x + x^2 + \cdots + x^{k-1})}
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     &= \frac{x^k + x^{k+1} + \cdots + x^n}{1 - (x + x^2 + \cdots + x^{k-1})} = \frac{x^k - x^{n+1}}{1 - 2x + x^k},
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\[
\begin{align*}
  u = \bar{k}_1 \ldots \bar{k}_r & \sim x^{k_1} \ldots x^{k_r} = x|u|, \\
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  [\bar{n}]^* & \sim (x + x^2 + \cdots + x^n)^* 
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Recall:

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The **norm** of $$u = \bar{k}_1 \ldots \bar{k}_r \in \mathbb{P}^*$$ is $$|u| = \sum_i k_i.$$  

Let $$x$$ be a variable and substitute $$\bar{k} \sim x^k.$$  

$$u = \bar{k}_1 \ldots \bar{k}_r \sim x^{k_1} \ldots x^{k_r} = x^{|u|},$$  

$$z(\bar{k}) \sim (x^k + x^{k+1} + \cdots + x^n)(x + x^2 + \cdots + x^{k-1})^* = \frac{x^{k} + x^{k+1} + \cdots + x^n}{1 - (x + x^2 + \cdots + x^{k-1})} = \frac{x^{k} - x^{n+1}}{1 - 2x + x^k},$$  

$$[\bar{n}]^* \sim (x + x^2 + \cdots + x^n)^* = \frac{1 - x}{1 - 2x + x^{n+1}}.$$  

The **type** of $$u \in [\bar{n}]^*$$ is $$t(u) = (t_1, \ldots, t_n)$$ where $$t_k = \# \text{ of } \bar{k} \in u.$$
Recall:

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Corollary (B & S)

If \( u \in [\bar{n}]^* \) has \( t(u) = (t_1, \ldots, t_n) \) then

\[
\sum_{w \geq u} x^{|w|} = \frac{1 - x}{1 - 2x + x^{n+1}} \prod_{k=1}^n \left( \frac{x^k - x^{n+1}}{1 - 2x + x^k} \right)^{t_k}.
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Note: 1. Letting \( n \to \infty \) in this corollary we get \( u \in \mathbb{P}^* \) and the \( x^{n+1} \) terms in the product drop out.
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\[
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Corollary (Björner & S)

If \( u \in [\bar{n}]^* \) has \( t(u) = (k_1, \ldots, k_n) \) then

\[
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\]

\[\text{Note: } 1. \text{ Letting } n \to \infty \text{ in this corollary we get } u \in \mathbb{P}^* \text{ and the } x^{n+1} \text{ terms in the product drop out. So}
\]

\[
\sum_{N \geq 0} c_N x^N = \sum_{w \geq \epsilon} x^{|w|} = \frac{1 - x}{1 - 2x} \cdot 1
\]

since \( t(\epsilon) = (0, 0, \ldots) \).
2. For any set $A$, define *subword order* on $A^*$ by: If $u = k_1 \ldots k_r$ and $w = l_1 \ldots l_s$ then $u \leq w$ iff there is $l_{i_1} \ldots l_{i_r}$ with

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**Theorem (Björner and Reutenauer)**

In subword order, $Z(u) = \sum_{w \geq u} w$ is rational.
2. For any set $A$, define **subword order** on $A^*$ by: If $u = k_1 \ldots k_r$ and $w = l_1 \ldots l_s$ then $u \leq w$ iff there is $l_i \ldots l_{ir}$ with

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**Theorem (Björner and Reutenauer)**

*In subword order, $Z(u) = \sum_{w \geq u} w$ is rational.*

For any poset $P$, define **generalized subword order** on $P^*$ by: If $u = k_1 \ldots k_r$ and $w = l_1 \ldots l_s$ then $u \leq_{P^*} w$ iff there is $l_i \ldots l_{ir}$ with

$$k_j \leq_P l_i \text{ for } 1 \leq j \leq r.$$
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$P$ an antichain $\Rightarrow$ $P^*$ is subword order,
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**Theorem (Björner and Reutenauer)**

In subword order, $Z(u) = \sum_{w \geq u} w$ is rational. □

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$P$ an antichain \hspace{1cm} $\Rightarrow$ \hspace{1cm} $P^*$ is subword order, 

$P$ a chain \hspace{1cm} $\Rightarrow$ \hspace{1cm} $P^*$ is composition order.
2. For any set $A$, define *subword order* on $A^*$ by: If $u = k_1 \ldots k_r$ and $w = l_1 \ldots l_s$ then $u \leq w$ iff there is $l_{i_1} \ldots l_{i_r}$ with

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Theorem (Björner and Reutenauer)
*In subword order, $Z(u) = \sum_{w \geq u} w$ is rational.*

For any poset $P$, define *generalized subword order* on $P^*$ by: If $u = k_1 \ldots k_r$ and $w = l_1 \ldots l_s$ then $u \leq_{P^*} w$ iff there is $l_{i_1} \ldots l_{i_r}$ with

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$P$ an antichain $\Rightarrow$ $P^*$ is subword order,

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Theorem (Björner & S)
*In generalized subword order, $Z(u) = \sum_{w \geq u} w$ is rational.*
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The *incidence algebra* of poset $P$ over the rationals $\mathbb{Q}$ is

$$I(P) = \{ \phi : P \times P \to \mathbb{Q} : \phi(u, w) = 0 \text{ if } u \nleq w \}.$$
The *incidence algebra* of poset $P$ over the rationals $\mathbb{Q}$ is

$$I(P) = \{\phi : P \times P \rightarrow \mathbb{Q} : \phi(u, w) = 0 \text{ if } u \not\leq w\}.$$ 

The *zeta function* is $\zeta \in I(P)$ defined by

$$\zeta(u, w) = \begin{cases} 
1 & \text{if } u \leq w, \\
0 & \text{else}.
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The *Möbius function* is $\mu \in I(P)$ defined by

$$\mu = \zeta^{-1}.$$
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**Question:** What is \( \mu \) in composition order on \( \mathbb{P}^* \)?
The *incidence algebra* of poset $P$ over the rationals $\mathbb{Q}$ is

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**Question:** What is $\mu$ in composition order on $P^*$? We first discuss $\mu$ in subword order on $A^*$.  

Suppose $0 \notin A$. An expansion of $u \in A^*$ is $\eta \in (A \cup \{0\})^*$ gotten by inserting zeros into $u$. 
Suppose $0 \not\in A$. An expansion of $u \in A^*$ is $\eta \in (A \cup \{0\})^*$ gotten by inserting zeros into $u$. An embedding $I$ of $u$ into $w$ corresponds to an expansion $\eta_u$: put $u$ in the positions of the $I$ and zeros elsewhere.
Suppose $0 \not\in A$. An *expansion* of $u \in A^*$ is $\eta \in (A \cup \{0\})^*$ gotten by inserting zeros into $u$. An embedding $I$ of $u$ into $w$ corresponds to an expansion $\eta_u$: put $u$ in the positions of the $I$ and zeros elsewhere.

**Ex.** If $A = \{a, b\}$, $u = a \ b \ b \ a$ and $w = a \ a \ b \ b \ b \ a \ b \ a$ then $w = a \ a \ b \ b \ b \ a \ b \ a$ corresponds to $\eta_u = 0 \ a \ 0 \ 0 \ b \ 0 \ b \ a$.
Suppose $0 \not\in A$. An expansion of $u \in A^*$ is $\eta \in (A \cup \{0\})^*$ gotten by inserting zeros into $u$. An embedding $I$ of $u$ into $w$ corresponds to an expansion $\eta_u$: put $u$ in the positions of the $I$ and zeros elsewhere. The support of $\eta_u$ is $\text{Supp} \eta_u = I$.

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**Ex.** If $A = \{a, b\}$, $u = a\ b\ b\ a$ and $w = a\ a\ b\ b\ b\ a\ b\ a$ then $w = a\ a\ b\ b\ b\ a\ b\ a$ corresponds to $\eta_u = 0\ a\ 0\ 0\ b\ 0\ b\ a$ and $I = \text{Supp} \eta_u = \{2, 5, 7, 8\}$. 
Suppose $0 \not\in A$. An expansion of $u \in A^*$ is $\eta \in (A \cup \{0\})^*$ gotten by inserting zeros into $u$. An embedding $l$ of $u$ into $w$ corresponds to an expansion $\eta_u$: put $u$ in the positions of the $l$ and zeros elsewhere. The support of $\eta_u$ is $\text{Supp} \eta_u = l$.

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A run of $k$’s in $w = k_1 \ldots k_t$ is a maximal interval $[r, s]$ with $k_r = k_{r+1} = \ldots = k_s$. 
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In $A^*$: $\mu(u, w) = (-1)^{\#w-\#u} (\# \text{ of normal } \eta_u \text{ in } w)$. ■

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So $\mu(u, w) = (-1)^{8-4} 2 = 2$. 


In $\mathbb{P}^*$, embedding $\eta_u = l_1 \ldots l_t$ of $u$ into $w = k_1 \ldots k_t$ is *normal* if

1. $l_i = k_i$, $k_i - 1$, or 0 for all $i$. 

\[ \sum_{\eta_u} (-1)^{d(\eta_u)} \] where the sum is over all normal embeddings $\eta_u$ into $w$.  

Ex. Suppose $u = 2 1 1 1 3$ and $w = 2 2 1 1 1 3 3$ abnormal $\eta_u$:  

$$
\begin{array}{cccccccc}
2 & 0 & 0 & 1 & 1 & 1 & 3 & 0 \\
2 & 0 & 1 & 1 & 1 & 3 & 0
\end{array}
$$

normal $\eta_u$:  

$$
\begin{array}{cccccccc}
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\[
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\text{abnormal } \eta_u &: 2 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 3 \\
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Given $\eta_u = k_1 \ldots k_t$ normal in $w = l_1 \ldots l_t$, it’s **defect** is

$$d(\eta_u) = \#\{ i \mid k_i = l_i - 1 \}.$$ 

**Ex.** Suppose $u = 21113$ and $w = 22111133$ and

<table>
<thead>
<tr>
<th>Abnormal $\eta_u$</th>
<th>Normal $\eta_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 0 0 1 1 1 3 3</td>
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</tr>
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In $\mathbb{P}^*$ we have

$$\mu(u, w) = \sum_{\eta_u} (-1)^{d(\eta_u)}$$

where the sum is over all normal embeddings $\eta_u$ into $w$.

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3. (Björner & S) using formal power series in noncommuting variables.
Outline

Compositions

Rational generating functions

Commuting variables

The zeta and Möbius functions

Comments and open problems
1. Is there a bijective proof that the norm generating function for compositions only depends on type? That is, given $u, u' \in \mathbb{P}^*$ with $t(u) = t(u')$, find a norm-preserving bijection

$$\{ w : w \geq u \} \leftrightarrow \{ w : w \geq u' \}.$$
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2. Björner and Reutenauer gave generating functions for the powers $\zeta^m$ for $m \geq 1$ in subword order on $A^*$. Björner and S were only able to do this for composition order on $[2]^*$, and the proof involved hypergeometric series identities. What can be said for $[n]^*$?
3. What can be said about $\mu$ in $P^*$ for an arbitrary poset $P$?
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Conjecture (Sagan & V)

For all $i \leq j$, the value $\mu(a^i, c^j)$ is the coefficient of $x^{j-i}$ in $T_{i+j}(x)$, the Tchebyshev polynomial of the first kind.
THANKS FOR LISTENING!