Monomial Bases for NBC Complexes

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1. Complexes and chromatic polynomials

Let Δ be a *simplicial complex* on a finite set E, so Δ is a family of subsets of E satisfying

$$S \in \Delta \text{ and } T \subseteq S \text{ implies } T \in \Delta.$$

The $S \in \Delta$ are called *faces*. We assume Δ is *pure of rank* r meaning that $|S| = r$ for all maximal faces $S \in \Delta$. For $0 \le i \le r$, let

$$f_i = f_i(\Delta) = \# \text{ of faces } S \in \Delta \text{ with } |S| = i.$$

The *f-polynomial* of Δ is

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_r x^r.$$

The *h-polynomial* of Δ is

$$\begin{aligned} h(x) &= (1-x)^r f\left(\frac{x}{1-x}\right) \\ &= f_0(1-x)^r + f_1 x(1-x)^{r-1} + \cdots + f_r x^r. \end{aligned}$$

and let

$$h_i = h_i(\Delta) = \text{coefficient of } x^i \text{ in } h(x).$$

Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. A *proper coloring* of G is $c : V \to \{1, 2, \ldots, \lambda\}$ such that

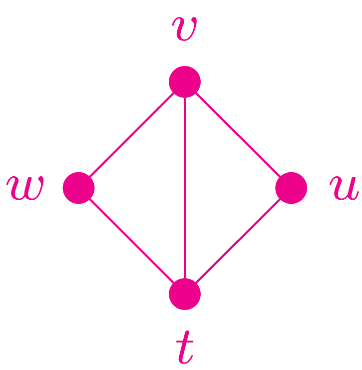
$$vw \in E \quad \text{implies} \quad c(v) \neq c(w).$$

The *chromatic polynomial* of G is

$$P(G) = P(G; \lambda) = \#\text{ of such proper colorings.}$$

Example. Let

$G =$



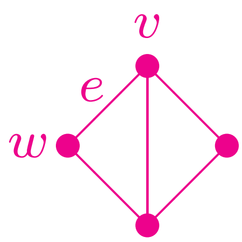
$$\begin{aligned} P(G) &= \#\text{ ways to color } t\text{, then } u\text{, then } v\text{, then } w \\ &= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 2) \\ &= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda. \end{aligned}$$

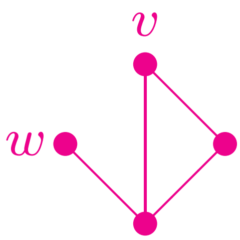
Proposition 1 *Let $\overline{K}_p$ be the edgeless graph and let T be a tree on p vertices. Then*

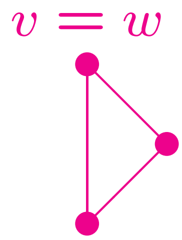
$$P(\overline{K}_p; \lambda) = \lambda^p \quad \textit{and} \quad P(T; \lambda) = \lambda(\lambda - 1)^{p-1}.$$

Let G be a graph and $e \in E$. Let

$$\begin{aligned} G \backslash e &= G \text{ with } e \text{ deleted,} \\ G/e &= G \text{ with } e \text{ contracted.} \end{aligned}$$

Example. Let $G =$ 

$G \backslash e =$ 

$G/e =$ 

Theorem 2 (Deletion-Contraction) *For $e \in E$*

$$P(G) = P(G \backslash e) - P(G/e)$$

Proof. If $e = vw$ then

$$\begin{aligned} P(G \backslash e) &= (\#\text{ proper } c \text{ for } G \backslash e \text{ s.t. } c(v) \neq c(w)) \\ &\quad +(\#\text{ proper } c \text{ for } G \backslash e \text{ s.t. } c(v) = c(w)) \\ &= P(G) + P(G/e). \qquad \blacksquare \end{aligned}$$

Corollary 3 *For any graph G:*

1. *$P(G;\lambda)$ is a monic polynomial in λ.*
2. *$\deg P(G;\lambda) = p = |V|$.*
3. *Coefficients of $P(G;\lambda)$ alternate in sign.*

2. NBC complexes

Define coefficients f_i by

$$P(G;\lambda) = f_0\lambda^p - f_1\lambda^{p-1} + \cdots$$

and coefficients h_i by

$$P(G;\lambda) = h_0\lambda(\lambda-1)^{p-1} - h_1\lambda(\lambda-1)^{p-2} + \cdots.$$

Let

$$\mathcal{C} = \mathcal{C}(G) = \text{set of cycles/circuits of } G.$$

Let G be *ordered* meaning that E has been given a total order $e_1 < e_2 < \ldots < e_q$. Then each $C \in \mathcal{C}$ has *broken circuit*

$$\overline{C} = C - \min C.$$

The *NBC complex* of G is

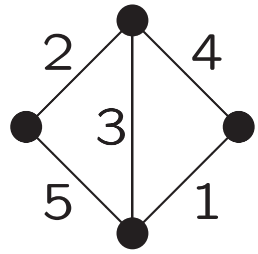
$$\Delta = \Delta(G) = \{S \subseteq E \ : \ S \text{ contains no } \overline{C}\}.$$

Then $\Delta(G)$ is a pure simplicial complex.

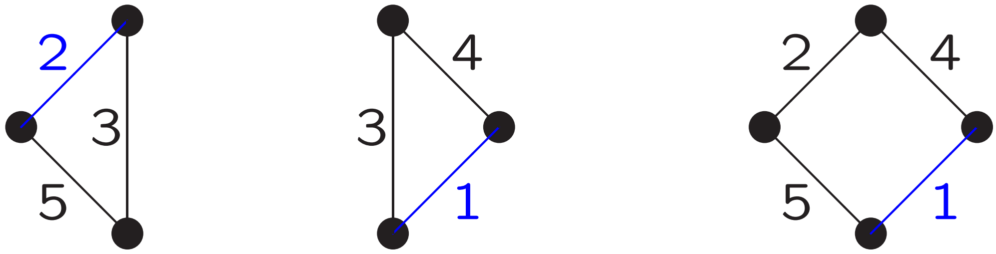
Theorem 4 *Let $P(G;\lambda)$ have coefficients f_i and h_i as defined above. Then for $0 \le i \le p$*

$$f_i = f_i(\Delta(G)) \quad \text{and} \quad h_i = h_i(\Delta(G)).$$

$G =$

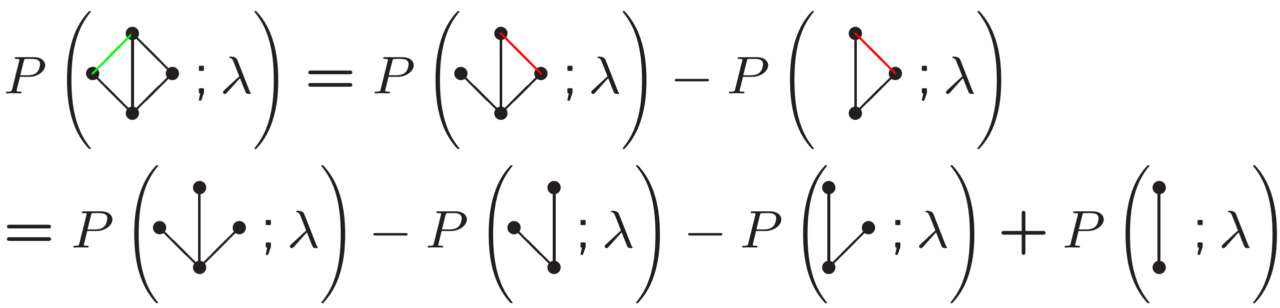


$\mathcal{C}(G) =$



$$
\begin{aligned}
\bar{\mathcal{C}}(G) &= \{35, 34, 245\} \\
\Delta(G) &= \{\emptyset\} \cup \{1, 2, 3, 4, 5\} \\
&\qquad \cup\{12, 13, 14, 15, 23, 24, 25, 45\} \\
&\qquad \cup\{123, 124, 125, 145\} \\
(f_i(\Delta)) &= (1, 5, 8, 4, 0).
\end{aligned}
$$

$$P(G; \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2 = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda.$$



$$
\begin{aligned}
&= \lambda(\lambda - 1)^3 - \lambda(\lambda - 1)^2 - \lambda(\lambda - 1)^2 + \lambda(\lambda - 1) \\
&= \lambda(\lambda - 1)^3 - 2\lambda(\lambda - 1)^2 + \lambda(\lambda - 1) \\
&(h_i(\Delta)) = (1, 2, 1, 0, 0)
\end{aligned}
$$

3. Stanley-Reisner rings and hsop's

Let $F[\mathbf{x}]$ be the polynomial ring over field F with variables $\mathbf{x} = \{x_1, \ldots, x_q\}$. If $E = \{e_1, \ldots, e_q\}$ then $S \subseteq E$ has *monomial*

$$\mathbf{x}^S = \prod_{e_i \in S} x_i.$$

Simplicial complex Δ has *Stanley-Reisner ring*

$$F(\Delta) = F[\mathbf{x}]/(\mathbf{x}^S \ : \ S \notin \Delta).$$

In particular, for an ordered graph G we let

$$F(G) = F(\Delta(G)) = F[\mathbf{x}]/(\mathbf{x}^{\overline{C}} \ : \ C \in \mathcal{C}(G)).$$

Now $F(G)$ has a *homogeneous system of parameters (hsop) of degree one* $\theta_1, \ldots, \theta_t$, i.e.,

1. θ_i is linear without constant term for all i,
2. $\theta_1, \ldots, \theta_t$ are algebraically independent,
3. $F(G)/(\theta_1, \ldots, \theta_t)$ is finite dim. over F.

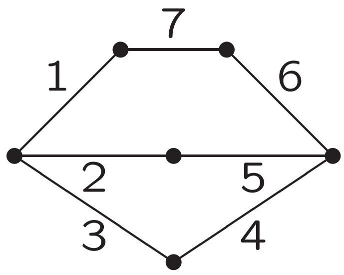
Brown gave an explicit hsop for $F(G)$. WLOG G is connected and let T be a spanning tree of G. If $e \in E(T)$ then e has *fundamental disconnecting set*

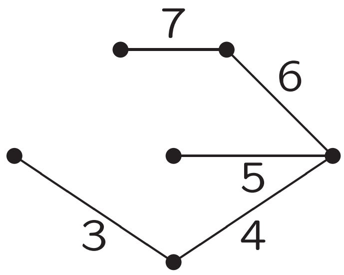
$$D_e = D_e(G) = \{f \in E(G) \ : \ T - e + f \text{ connected}\}$$

and hsop element (when $F = \mathbb{Z}_2$)

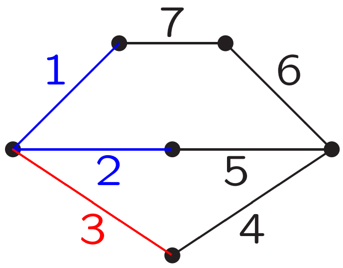
$$\theta_e = \sum_{e_i \in D_e} x_i.$$

Example

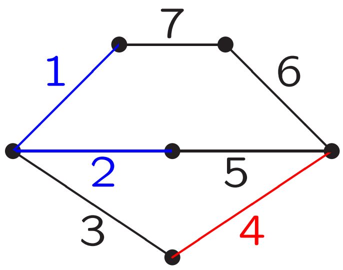
$G =$ 

$T =$ 

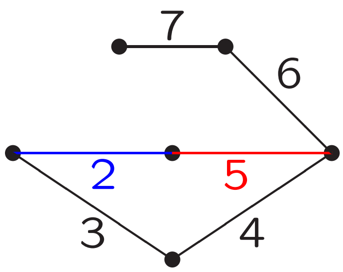
$$\begin{aligned}\mathcal{C}(G) &= \{\color{blue}{1}\color{black}{3467}, \color{blue}{2}\color{black}{345}, \color{blue}{1}\color{black}{2567}\} \\ \mathbb{Z}_2(G) &= \mathbb{Z}_2[x_1, \ldots, x_7]/(x_3x_4x_6x_7, x_3x_4x_5, x_2x_5x_6x_7)\end{aligned}$$



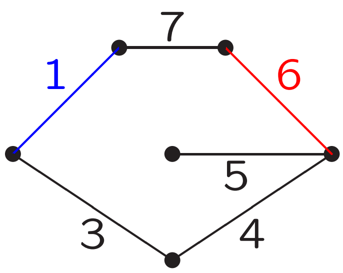
$$\theta_3 = \color{red}{x_3} \color{black}{+} \color{blue}{x_1} \color{black}{+} \color{blue}{x_2}$$



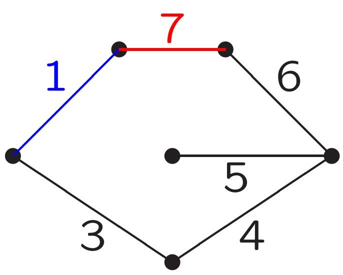
$$\theta_4 = \color{red}{x_4} \color{black}{+} \color{blue}{x_1} \color{black}{+} \color{blue}{x_2}$$



$$\theta_5 = \color{red}{x_5} \color{black}{+} \color{blue}{x_2}$$



$$\theta_6 = \color{red}{x_6} \color{black}{+} \color{blue}{x_1}$$



$$\theta_7 = \color{red}{x_7} \color{black}{+} \color{blue}{x_1}$$

4. Monomial ideals

If $F(\Delta)$ has an hsop $\theta_1, \ldots, \theta_t$ we let

$$R(\Delta) = F(\Delta)/(\theta_1, \ldots, \theta_t).$$

Consider

$$\text{Mon}(k) = \text{set of monomials in } F[x_1, \ldots, x_k].$$

A subset $L \subseteq \text{Mon}(k)$ is a *lower order ideal* if

$$m \in L \text{ and } n|m \text{ imples } n \in L.$$

The *lower order ideal generated* by $S \subseteq \text{Mon}(k)$ is

$$L(S) = \{m \in \text{Mon}(k) \ : \ m|n \text{ for some } n \in S\}.$$

Upper ideal and $U(S)$ are defined dually.

Theorem 5 (Macaulay, Stanley) *Suppose Δ is a simplicial complex and that the ring $F(\Delta)$ is Cohen-Macaulay. Then $R(\Delta)$ has a basis, L, which is a lower order ideal of monomials and*

$$h_i(\Delta) = \#\text{ of monomials of total degree } i \text{ in } L.$$

For a graph G, $F(G)$ is Cohen-Macaulay. We have a conjectured construction of a basis for $R(G)$.

An ordering $e_1 < \ldots < e_q$ is *standard* if the last $p-1$ edges form a tree. Let $k = |E(G) - E(T)|$. We can pick the monomial basis for $R(G)$ inside Mon(k) since Brown's θ_i can be used to eliminate the other variables, replacing each $\mathbf{x}^{\overline{C}}$ by a polynomial $p_{\overline{C}}$.

Example. In our running example, $k = 2$ and

$$\mathbb{Z}_2(G) = \mathbb{Z}_2[x_1, \ldots, x_7]/(x_3x_4x_6x_7, x_3x_4x_5, x_2x_5x_6x_7).$$
$$\theta_3 = x_3 + x_1 + x_2, \ \theta_4 = x_4 + x_1 + x_2, \ \theta_5 = x_5 + x_2.$$

So, picking one of the broken circuit monomials

$$\mathbf{x}^{\overline{C}} = x_3x_4x_5 \quad \text{becomes} \quad p_{\overline{C}} = (x_1 + x_2)^2 x_2.$$

For $1 \le i \le k$, the graph $T + e_i$ has a unique *fundamental circuit* C_i.

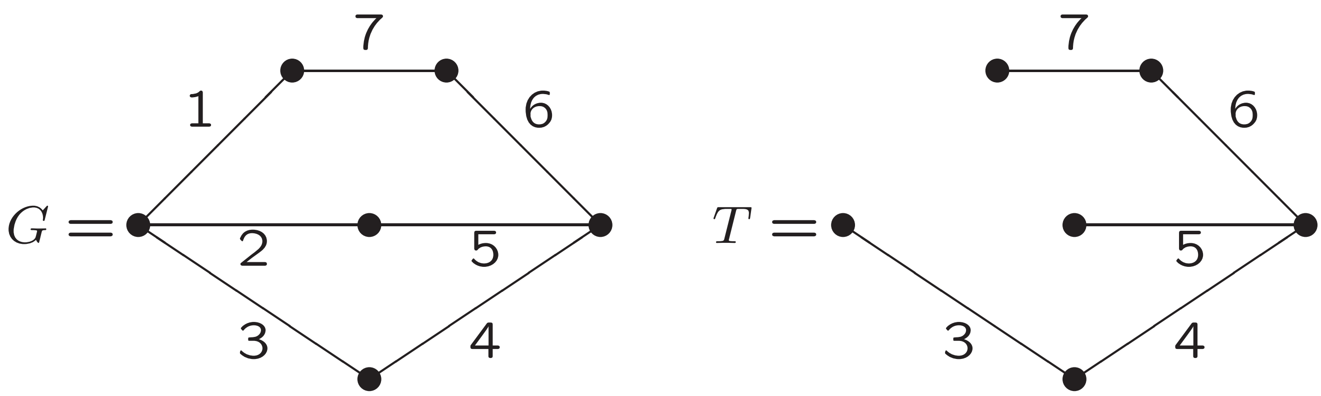
Conjecture 6 *Let G be connected. Then there is a standard ordering of E such that $R(G)$ has basis*

$$L(G) = \text{Mon}(k) - U(m_{\overline{C}} \ : \ C \in \mathcal{C}(G))$$

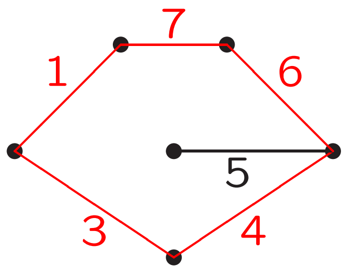
where

$$m_{\overline{C}} = \begin{cases} x_i^{\#\overline{C}_i} & \text{if } C = C_i \text{ fundamental,} \\ \min p_{\overline{C}} & \text{else.} \end{cases}$$

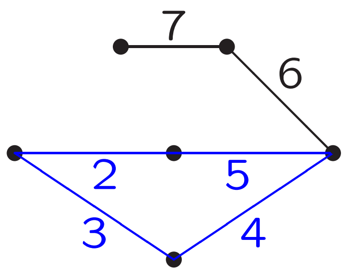
Here $\min p$ *picks out the lexicographically smallest monomial in* p.



Fundamental cycles:

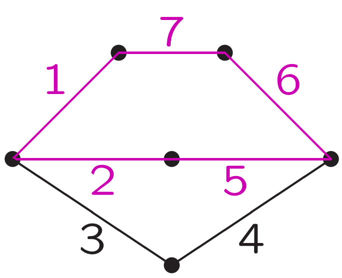


$C_1 = \{1, 3, 4, 6, 7\} \qquad m_{\overline{C}_1} = x_1^4$



$C_2 = \{2, 3, 4, 5\} \qquad m_{\overline{C}_2} = x_2^3$

Nonfundamental cycle:



$$\begin{aligned} C_3 &= \{1, 2, 5, 6, 7\} \qquad m_{\overline{C}_3} = x_1^2 x_2^2 \\ \mathbf{x}^{\overline{C}_3} &= x_2 x_5 x_6 x_7 \\ p_{\overline{C}_3} &= x_2 x_2 x_1 x_1 \end{aligned}$$

So $R(G)$ has basis

$$L(G) = \mathrm{Mon}(2) - U(x_1^4,\ x_2^3,\ x_1^2 x_2^2).$$

5. Comments

A graph with a standard ordering satisfying the conjecture is said to have a *broken circuit basis*.

a. (Generalized) theta graphs and phi graphs have broken circuit bases.

b. By only considering the fundamental circuits:

Proposition 7 *If G has a broken circuit basis and $c_i = |C_i|$ for $1 \leq i \leq k$, then $R(G)$ is spanned by*

$$L\left(\prod_{1\leq i\leq k} x_i^{c_i-2}\right).$$

Stanley showed that the number of acyclic orientations of G is given by $P(G;-1)$. So one can use this proposition to estimate their number.

c. The results we have about broken circuit bases are proved by deletion/contraction. It is hoped that together with the ear decomposition of a block we will be able to prove the full conjecture.

d. The conjecture may even be true for representable matroids.