The Möbius function of generalized subword order

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Outline

The Möbius function of ordinary subword order

Pattern order and generalized subword order

The Möbius function of generalized subword order

Open Questions

Let P be a finite poset (partially ordered set). The set of closed intervals of P is

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The Möbius function of P, $\mu : \operatorname{Int} P \to \mathbb{Z}$, is defined recursively by

$$\sum_{z \in [x,y]} \mu(x, z) = \delta_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{else.} \end{cases}$$

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The Möbius function is an important invariant of any poset.

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Outline

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Ex. We have that $\sigma = 321|4|65$ is layered with $\ell(\sigma) = (3, 1, 2)$.
Also $\pi = 21|43$ is layered and $\pi \leq \sigma$ since $\ell(\pi) = (2, 2)$ and $(2, 0, 2) \leq (3, 1, 2)$ component-wise.

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Let P be a rooted forest. If $v \leq w$ in P^* then

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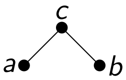
The smallest poset which is not a rooted forest is $\Lambda =$ 

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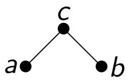
The smallest poset which is not a rooted forest is $\Lambda =$ 

Theorem (conjecture: S. and Vatter, proof: Tomie)

If $0 \leq i \leq j$, then in Λ^* we have

$$\mu(a^i, c^j) = \text{coefficient of } x^{j-i} \text{ in } T_{i+j}(x)$$

where $T_n(x)$ is the nth Tchebyshev polynomial of the 1st kind.

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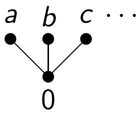
Now $A_0 =$ 

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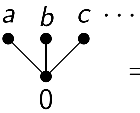
Now $A_0 =$ 

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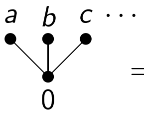
Now $A_0 =$ 

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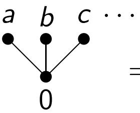
Now $A_0 =$ 

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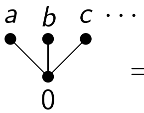
Now $A_0 =$ 

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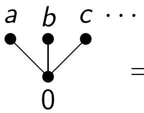
Now $A_0 =$ 

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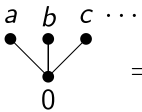
Now $A_0 =$ 

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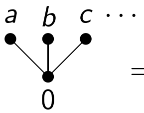
Now $A_0 =$ 

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Theorem (McNamara and S)

Let P be a poset with $\operatorname{rk} P \leq 1$. If $v < w$ then $\Delta(v, w)$ is homotopic to a wedge of $|\mu(v, w)|$ spheres all of dimension $|w| - |v| - 2$.

Outline

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THANKS FOR LISTENING!