The Möbius function of generalized subword order

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Pattern order and generalized subword order

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Open Questions
Outline

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Open Questions
Let $P$ be a finite poset (partially ordered set). The set of closed intervals of $P$ is

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The Möbius function of $P$, $\mu : \text{Int } P \to \mathbb{Z}$, is defined recursively by

$$\sum_{z \in [x, y]} \mu(x, z) = \delta_{x, y} = \begin{cases} 
1 & \text{if } x = y, \\
0 & \text{else.}
\end{cases}$$
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The Möbius function is an important invariant of any poset.
Set $A$ has a **Kleene closure** consisting of all finite words over $A$:

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A **subword** of $w$ is $v = w(i_1) \ldots w(i_k)$ with $i_1 < \cdots < i_k$. 

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Subword order on $A^*$ has $v \leq w$ iff $v$ is a subword of $w$. 

Suppose $0 \not\in A$. An embedding of $v$ in $w$ is $\eta \in (A \cup \{0\})^*$ gotten by zeroing out letters of $w$ and leaving $v$. 

∴ $v \leq w$ in $A^*$ iff there is an embedding of $v$ in $w$. 

A run in $w$ is a maximal consecutive subword with all elements equal. 

Normal embeddings can only zero out a letter if it is first in a run (but not all such letters must be made zero). 

Theorem (Björner) If $|w|$ denotes the length of $w$ and $v \leq w$ in $A^*$ then 

$$\mu(v, w) = (-1)^{|w| - |v|} \cdot \# \text{ of normal embeddings of } v \text{ in } w.$$ 

Ex. $w = aabba$ has runs $aa$, $bb$, and $a$. 

The only normal embedding of $v = aba$ is $0a0ba$. 

∴ $\mu(v, w) = (-1)^5 - 3 \cdot 1 = 1$. 


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Open Questions
Call sequences of distinct integers \( \pi = \pi(1) \ldots \pi(k) \) and 
\( \sigma = \sigma(1) \ldots \sigma(k) \) order isomorphic, \( \pi \cong \sigma \), if 

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\pi(i) < \pi(j) \iff \sigma(i) < \sigma(j) \text{ for all } i, j.
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Let \( \mathcal{S}_n \) be the symmetric group on \( \{1, \ldots, n\} \) and let \( \mathcal{S} = \bigcup_n \mathcal{S}_n \).

Say \( \sigma \in \mathcal{S}_n \) contains pattern \( \pi \in \mathcal{S}_k \) if \( \sigma \) has a subword \( \sigma' \cong \pi \).
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**Ex.** $132 \cong 475$ so $\sigma = 6437125$ contains $\pi = 132$.

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**Question (Wilf)**

*What is the Möbius function of $\mathcal{S}$?*
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Let $\mathcal{S}_n$ be the *symmetric group* on $\{1, \ldots, n\}$ and let $\mathcal{S} = \bigcup_n \mathcal{S}_n$. Say $\sigma \in \mathcal{S}_n$ contains pattern $\pi \in \mathcal{S}_k$ if $\sigma$ has a subword $\sigma' \cong \pi$. Pattern order on $\mathcal{S}$ is $\pi \leq \sigma$ iff $\sigma$ contains $\pi$.

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Call $\sigma \in \mathcal{S}$ *layered* with layer lengths $\ell(\sigma) = (k, l, \ldots)$ if

$$\sigma = k(k - 1) \ldots 1(k + l)(k + l - 1) \ldots k + 1 \ldots.$$
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**Ex.** We have that $\sigma = 321|4|65$ is layered with $\ell(\sigma) = (3, 1, 2)$. 
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Also $\pi = 21|43$ is layered and $\pi \leq \sigma$ since $\ell(\pi) = (2, 2)$ and $(2, 0, 2) \leq (3, 1, 2)$ component-wise.
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One can generalize embeddings and normal embeddings to $P^*$. 

Theorem (S. and Vatter)

Let $P$ be a rooted forest. If $v \leq w$ in $P^*$ then

$$\mu(v, w) = \sum \eta(-1)^{d(\eta)}$$

where the sum is over all normal embeddings $\eta$ of $v$ in $w$, and $d(\eta)$ is the number of indices $i$ where $w(i)$ covers $\eta(i)$.

Theorem (conjecture: S. and Vatter, proof: Tomie)

If $0 \leq i \leq j$, then in $\Lambda^*$ we have

$$\mu(a_i, c_j) = \text{coefficient of } x^{j-i} \text{ in } T_{i+j}(x)$$

where $T_n(x)$ is the $n$th Tchebyshev polynomial of the 1st kind.
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$$\mu(v, w) = \sum_{\eta} (-1)^{d(\eta)}$$

where the sum is over all normal embeddings $\eta$ of $v$ in $w$, and $d(\eta)$ is the number of indices $i$ where $w(i)$ covers $\eta(i)$. 
Let $P$ be any poset. Generalized subword order on $P^*$ has $v \leq w$ iff there is a subword $w(i_1) \ldots w(i_k)$ of length $|v|$ with
\[v(1) \leq_P w(i_1), \ldots, v(k) \leq_P w(i_k).\]

**Ex.**
1. $P = A$ an antichain $\implies P^* \cong A^*$ (ordinary subword).
2. $P = \mathbb{P}$ (positive integers) $\implies P^* \cong \mathcal{L}$ (layered).

One can generalize embeddings and normal embeddings to $P^*$.

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**Theorem (S. and Vatter)**

Let \( P \) be a rooted forest. If \( v \leq w \) in \( P^* \) then
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**Theorem (conjecture: S. and Vatter, proof: Tomie)**

If \( 0 \leq i \leq j \), then in \( \Lambda^* \) we have
\[
\mu(a^i, c^j) = \text{coefficient of } x^{j-i} \text{ in } T_{i+j}(x)
\]
where \( T_n(x) \) is the \( n \)th Tchebyshev polynomial of the 1st kind.
Outline

The Möbius function of ordinary subword order

Pattern order and generalized subword order

The Möbius function of generalized subword order

Open Questions
If $P$ is any poset then let $P_0$ be $P$ with a new minimum element 0.
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The proof uses discrete Morse theory and classical results about $\mu$. 

If $P$ is any poset then let $P_0$ be $P$ with a new minimum element 0. Let $\mu_0$ be the Möbius function of $P_0$. Call $P$ locally finite if $\# [a, b]$ finite for all $a \leq b$ in $P$. 

Theorem (McNamara and S.) Let $P$ be a poset such that $P_0$ is locally finite. Then 

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\mu(v, w) = \sum_{\eta \mid w} \prod_{i=1}^{\eta(0)} \left\{ \mu_0(\eta(i), w(i)) + 1 \text{ if } \eta(i) = 0 \text{ and } w(i-1) = w(i), \right. \\
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\mu_0(\eta(i), w(i)) + 1 & \text{if } \eta(i) = 0 \text{ and } w(i - 1) = w(i), \\
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**Corollary**

*If A is an antichain then in A*\(^*\)

\[ \mu(v, w) = (-1)^{\lvert w \rvert - \lvert v \rvert} (\# \text{ of normal embeddings of } v \text{ in } w). \]
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Proof Claim \( \eta \) not normal iff \((\exists i : \eta(i) = 0 \text{ and } w(i - 1) = w(i))\).
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Now \( A_0 = \) [Diagram of a tree with nodes labeled a, b, c, ... and a root labeled 0]
\[
\mu(v, w) = \sum_{\eta} \prod_{i=1}^{\text{\mid w\mid}} \begin{cases} 
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Now \( A_0 = \begin{array}{c}
\text{a b c }\
\text{0}
\end{array} \implies \mu_0(\eta(i), w(i)) = \begin{cases} 
+1 & \text{if } \eta(i) = w(i), \\
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\mu(v, w) = (-1)^{|w|-|v|} \left( \# \text{ of normal embeddings of } v \text{ in } w \right).
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Now \( A_0 = \begin{array}{c}
0 \\
\bullet \\
\bullet \\
\bullet \\
a \\
b \\
c \\
\cdots
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\begin{array}{ccc}
a & b & c \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \)  
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0 & \quad \bullet & \quad \bullet & \quad \cdots
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\bullet \\
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**Proposition**

*If $(x, y)$ is finite then*

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\mu(x, y) = \tilde{\chi}(\Delta(x, y))
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**Theorem (McNamara and S)**

Let $P$ be a poset with $\text{rk} P \leq 1$. If $v < w$ then $\Delta(v, w)$ is homotopic to a wedge of $|\mu(v, w)|$ spheres all of dimension $|w| - |v| - 2$. 
Outline

The Möbius function of ordinary subword order

Pattern order and generalized subword order

The Möbius function of generalized subword order

Open Questions
1. What can be said about the Möbius function of other intervals in $\mathcal{G}$ (pattern order)?
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2. **Ordinary factor order** is given by $v \leq w$ if $v$ is a subword of consecutive letters in $w$. Björner determined the Möbius function of ordinary factor order.
1. What can be said about the Möbius function of other intervals in $\mathcal{S}$ (pattern order)? There has been recent work by Tenner-Steingrímsson and by Burstein-Jelínek-Jelínková-Steingrímsson.

2. Ordinary factor order is given by $v \leq w$ if $v$ is a subword of consecutive letters in $w$. Björner determined the Möbius function of ordinary factor order. Generalized factor order on $P^*$ for any poset $P$ can be defined analogously. Willenbring generalized Björner’s result to rooted trees. Is there a formula for any $P$?
1. What can be said about the Möbius function of other intervals in $S$ (pattern order)? There has been recent work by Tenner-Steingrímsson and by Burstein-Jelínek-Jelínkova-Steingrímsson.

2. **Ordinary factor order** is given by $v \leq w$ if $v$ is a subword of consecutive letters in $w$. Björner determined the Möbius function of ordinary factor order. **Generalized factor order** on $P^*$ for any poset $P$ can be defined analogously. Willenbring generalized Björner’s result to rooted trees. Is there a formula for any $P$? Note that Bernini-Ferrari-Steingrímsson determined the Möbius function of the consecutive pattern poset and S-Willenbring showed that there is an intimate connection between this poset and ordinary factor order.
THANKS FOR LISTENING!