Factoring the Characteristic Polynomial

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Outline

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Let P, Q be ranked posets.
1. $P \cong Q \implies \chi(P;t) = \chi(Q;t)$.
2. $P \times Q$ is ranked and $\chi(P \times Q;t) = \chi(P;t)\chi(Q;t)$.

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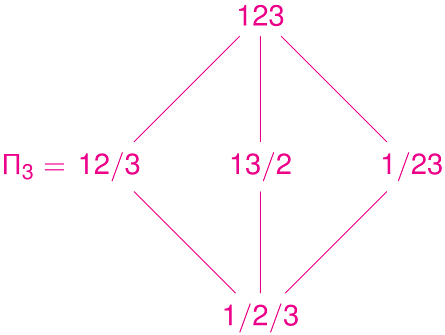

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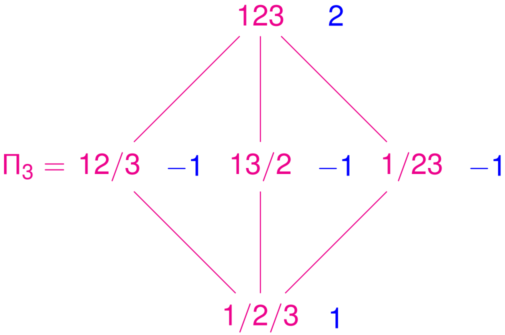

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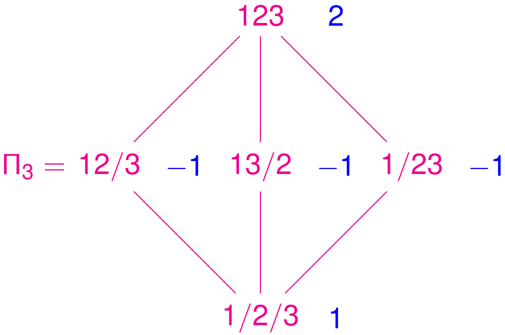

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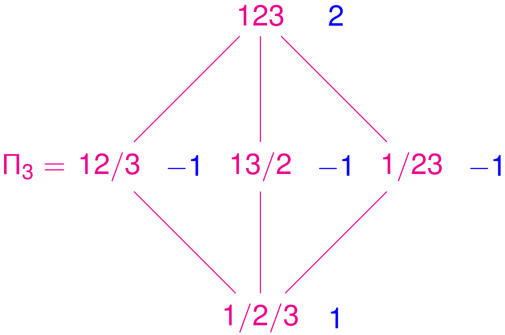

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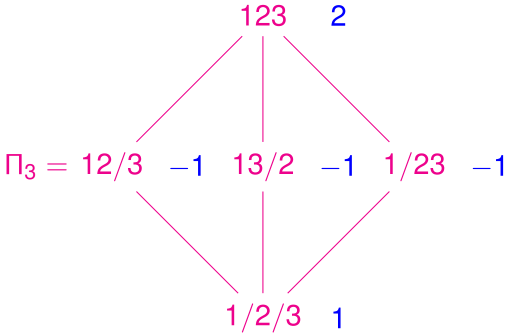

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\begin{aligned}
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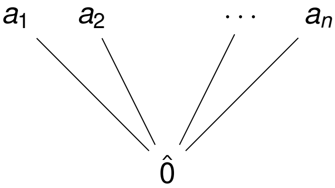

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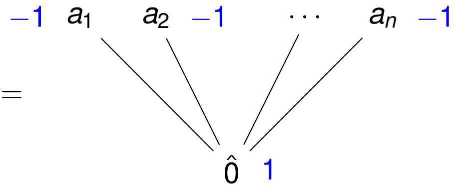

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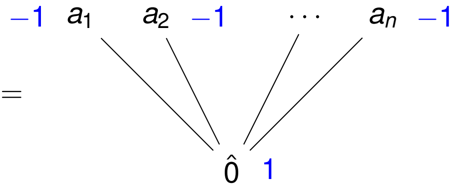

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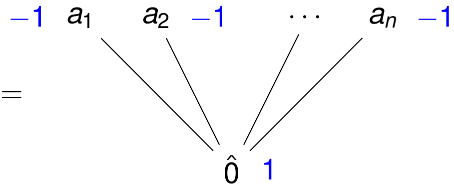

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So the characteristic polynomial of CL_n can give us any positive integer root as n varies.

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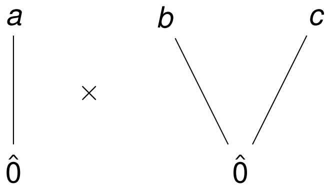

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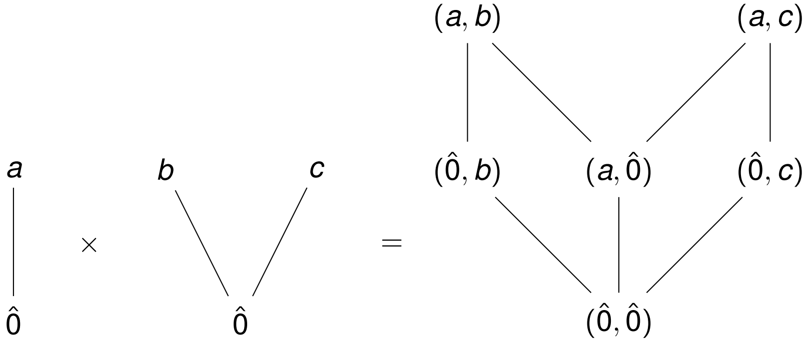

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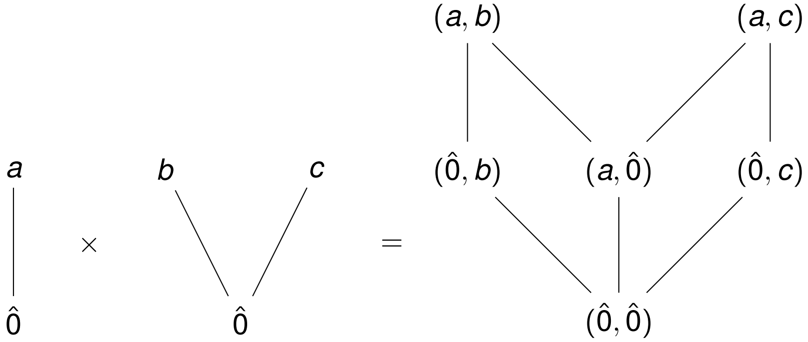

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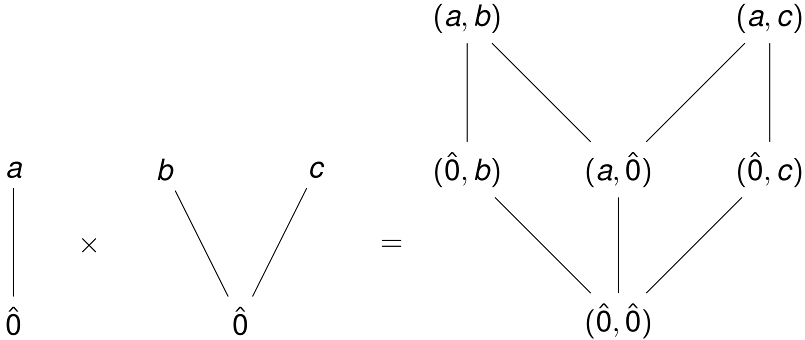

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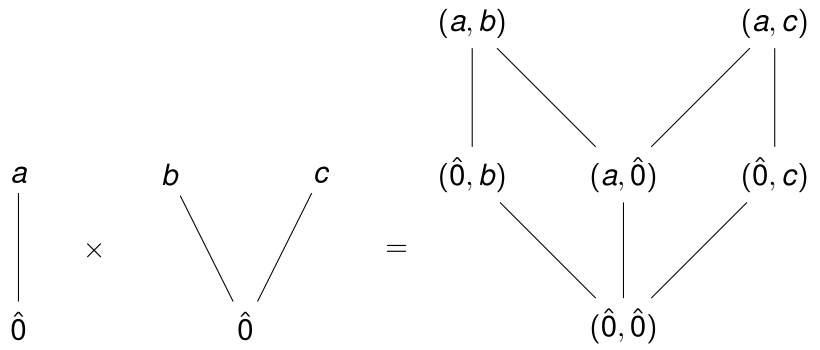

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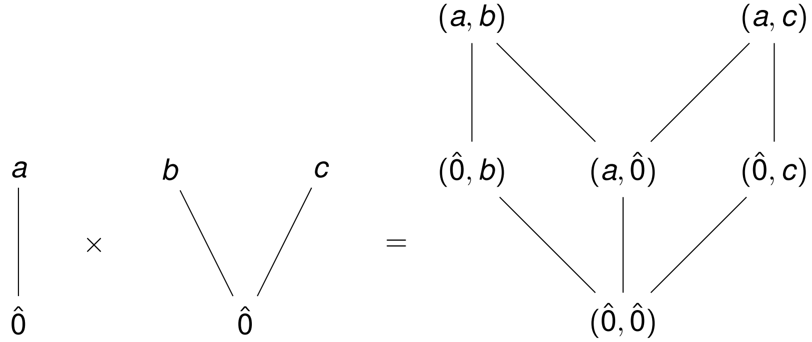

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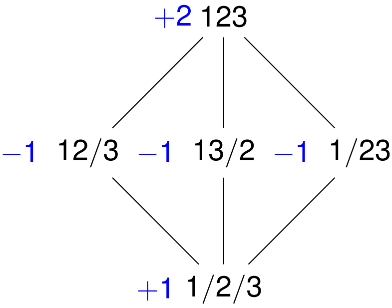

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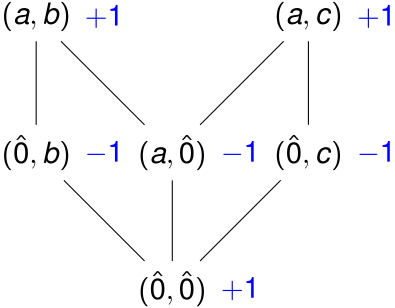

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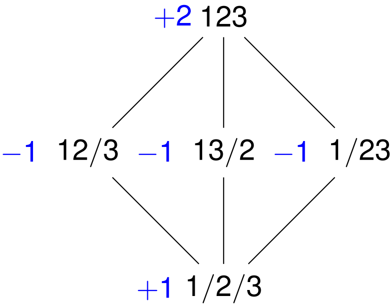

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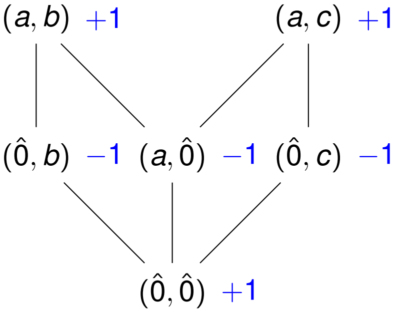

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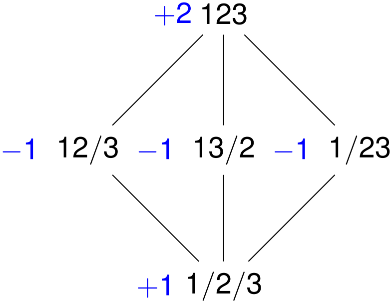

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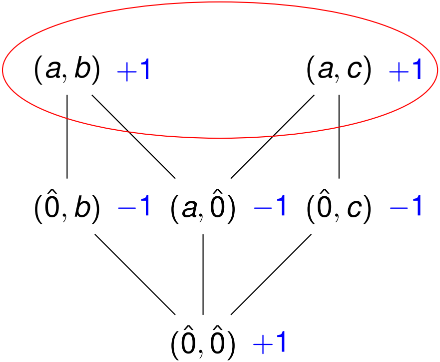

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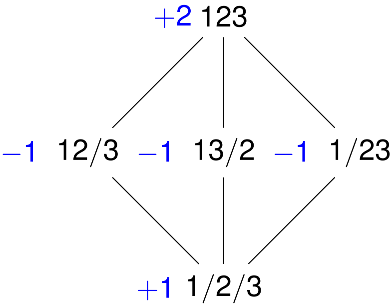

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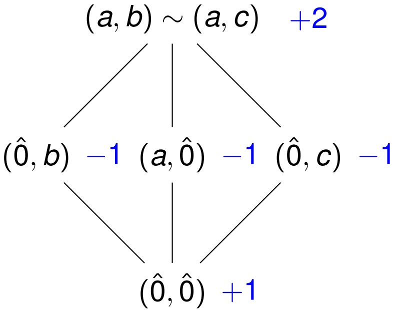

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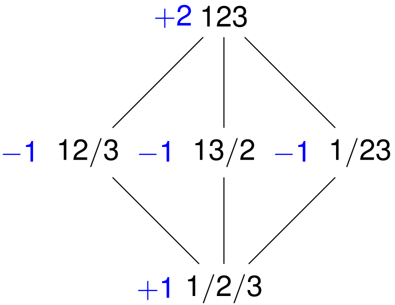

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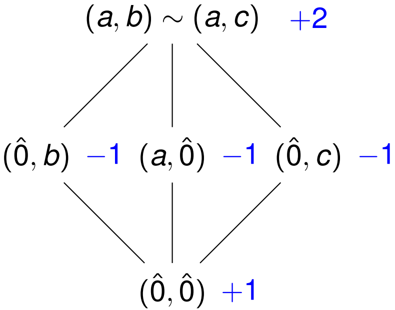

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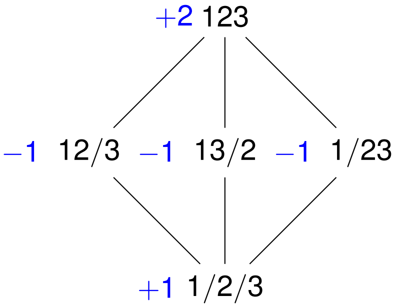

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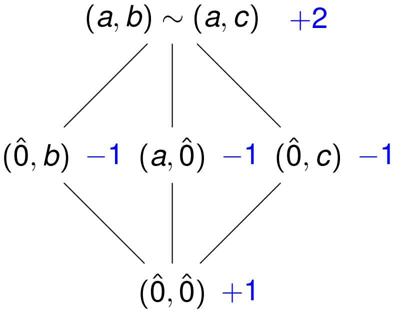

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Note that the Möbius values of (a, b) and (a, c) added to give the Möbius value of $(a, b) \sim (a, c)$. So $\chi(CL_1 \times CL_2)$ did not change after the identification since characteristic polynomials only record the sums of the Möbius values at each rank.

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$$X \leq Y \text{ in } P/\sim \iff x \leq y \text{ in } P \text{ for some } x \in X \text{ and some } y \in Y.$$

Quotients of posets *need not* be posets.

Ex. Consider

$$C_2 = \begin{array}{c} 2 \\ | \\ 1 \\ | \\ 0 \end{array}$$

Put an equivalence relation on C_2 with classes

$$X = \{0, 2\}, \qquad Y = \{1\}.$$

Then $X < Y$ since $0 < 1$ and $Y < X$ since $1 < 2$.

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Lemma (Hallam-S)

If $P/\sim$ is a homogeneous quotient then $P/\sim$ a poset.

Outline

How do we determine a suitable equivalence relation?

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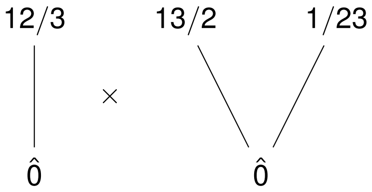

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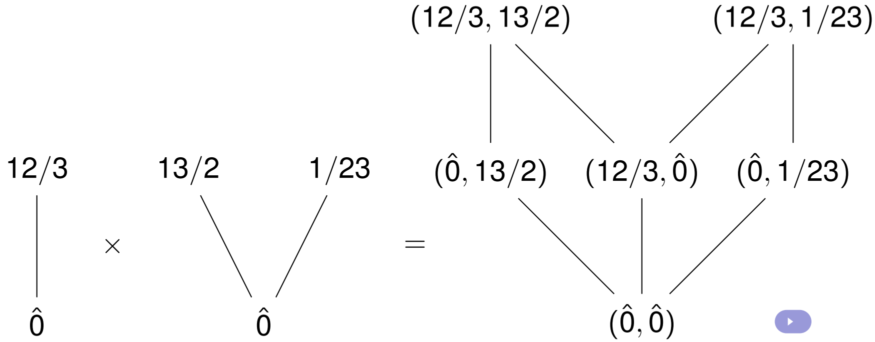

Now relabel each element of the product with the join of its two coordinates.

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\begin{tikzcd}
(12/3, 13/2) \arrow[d, dash] \arrow[dr, dash] & & (12/3, 1/23) \arrow[dl, dash] \arrow[d, dash] \\
(\hat{0}, 13/2) \arrow[dr, dash] & (12/3, \hat{0}) \arrow[d, dash] & (\hat{0}, 1/23) \arrow[dl, dash] \\
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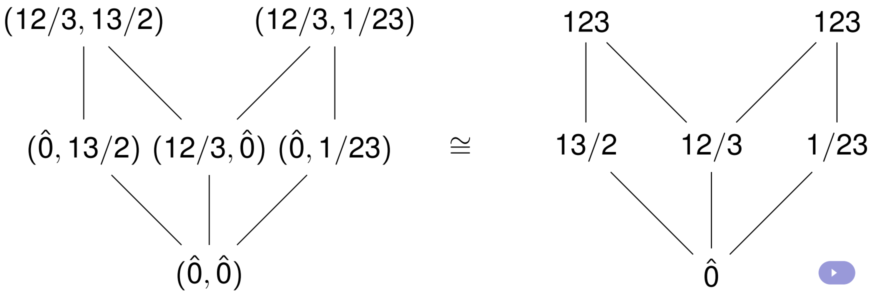

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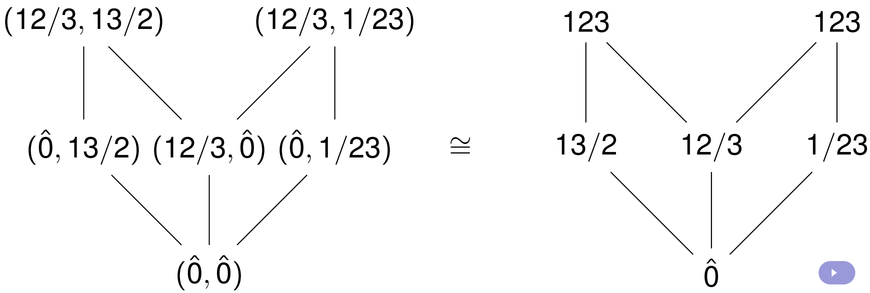

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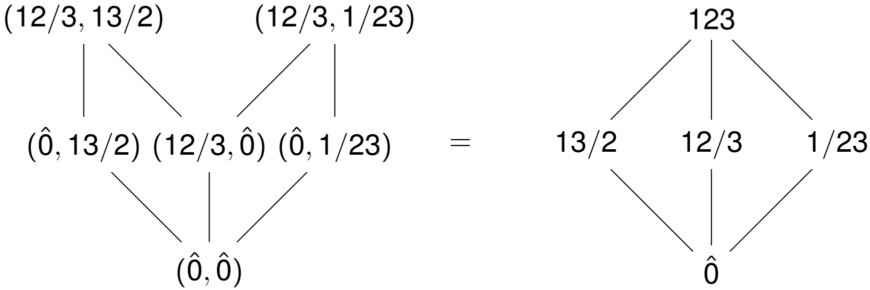

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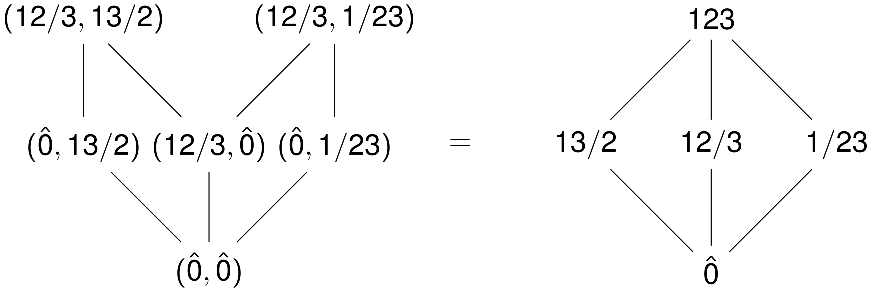

Finally, identify elements with the same label to obtain the same quotient we did before. Not only is the quotient isomorphic to Π_3, it even has the same labeling.

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The *atomic transversals of $x \in L$* are the elements of the equivalence class

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We need a condition on the standard equivalence relation which will make sure that the quotient is homogeneous and ranked.

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Let L be a lattice, $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$ and $Q = \prod_i CL_{A_i}$. Suppose that for all $x \in L$ and all $\mathbf{t} \in \mathcal{T}_x^a$ we have

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Then the standard equivalence relation is homogeneous, $Q/\sim$ is ranked, and

$$\rho(\mathcal{T}_x^a) = \rho(x).$$

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Condition (1) is used to prove that the map $(Q/\sim) \to L$ by $\mathcal{T}_x^a \mapsto x$ is surjective.

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Outline

Motivating Example

Quotient Posets

The Standard Equivalence Relation

The Main Theorem

Partitions Induced by Chains

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When will the partition induced by such a chain give the roots of a factorization? For $x \in L$ with $x \neq \hat{0}$, let i be the index with $x \leq x_i$ and $x \not\leq x_{i-1}$. Say that C satisfies the *meet condition* if, for every $x \in L$ of rank at least 2,

$$x \wedge x_{i-1} \neq \hat{0}.$$

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1. *For each $x \neq \hat{0}$ in L, there is i such that $|A_x \cap A_i| = 1$.*
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3. *The characteristic polynomial of L factors as*

$$\chi(L, t) = t^{\rho(L)-n} \prod_{i=1}^{n} (t - |A_i|).$$

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Corollary (Stanley, 1972)

Let L be a semimodular, supersolvable lattice and $(A_1, \ldots, A_n)$ be induced by a saturated chain of left-modular elements. Then

$$\chi(L; t) = \prod_{i=1}^{n} (t - |A_i|).$$

DEAR RICHARD: